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Monotonicity of likelihood support bounds for system failure rates

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Abstract

Simple expressions are derived for lower and upper support limits for the system failure probability of a series or parallel system based on binomial failure data. These limits are shown to be non-decreasing functions of the number of failures of any component for a fixed number of tests and non-increasing functions of the number of tests of any component for a fixed number of failures. This logically essential property, not shared by many other approximate upper limits, is critical to the efficient computation of Buehler bounds, using support limits as an ordering function.

Keywords: Buehler bounds; Profile likelihood; Exact confidence limits

1. Introduction

Setting confidence limits on the probability of failure of a system based on binomial test data of the components is an important problem in reliability theory; see for instance Winterbottom (1974, 1984) and Tian (2002) for a recent review. The difficulties lie in the discreteness of the sample space and the extremely poor coverage properties of approximate methods under the natural condition that the component failures probabilities are all small. A seminal paper by Buehler (1957) demonstrated how to construct optimal upper limits for the failure probability of a

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parallel system and it is the computation of these so-called Buehler bounds that partly motivate this work. These bounds require that a so-called designated statistic be chosen to order the sample space. Buehler upper/lower bounds are then as small/large as possible subject to this ordering and the coverage level chosen.

Buehler bounds and their optimality were generalised and formalised by Jobe and David (1992), Kabaila and Lloyd (1997) and Lloyd and Kabaila (2003). There are several aspects of Buehler bounds that still require clarification. First, Buehler bounds have been shown to have some unusual logical properties (see Reiser and Jaeger (1991), Harris and Soms (1991) and Kabaila and Lloyd (2004)). For instance, with certain designated statistics Buehler bounds for the failure probability can increase when a single success is added to the data. Second, Buehler bounds can be very expensive to compute, again depending on the choice of designated statistic. This has motivated intervals based on the maximum likelihood estimate (MLE) as a designated statistic, see Soms (1989). Thirdly, the wrong choice of designated statistic, in particular the MLE, can lead to bounds that have poor efficiency, for instance upper bounds that tend to be larger on average than with alternative choices. A large numerical study reported in Kabaila and Lloyd (2002) demonstrated that designated statistics based on the approximate chi-square distribution of the profile likelihood function perform better than other suggestions in terms of efficiency.

Our general framework is as follows. We have k components in a system that fail independently with probability p_i leading to system failure probability $\theta(p)$, where $p = (p_1, \ldots, p_k)$. The data comprise independent failure counts $y = (y_1, \ldots, y_k)$ out of $n = (n_1, \ldots, n_k)$ trials on each of the k components. The distribution of y_i is binomial(n_i, p_i). The likelihood function is $L(p) \propto \prod_{i=1}^{k} p_i^{y_i}(1-p_i)^{n_i-y_i}$, the MLE is $\hat{\theta}(y,n) = \theta(\hat{p})$ where $\hat{p}_i = y_i/n_i$ and the profile likelihood function for θ is

$$\mathscr{L}(t) = \frac{\sup_{\{\theta(p)=t\}} L(p)}{\sup_p L(p)}$$

where p ranges over the unit hypercube. The upper support limit $u_r(y,n)$ for θ is the value of $\theta \geqslant \hat{\theta}$ for which $\mathscr{L}(\theta) = r$. The lower support limit $l_r(y,n)$ for θ is the value of $\theta \leqslant \hat{\theta}$ for which $\mathscr{L}(\theta) = r$.

Under the weak assumption that $\theta(p)$ is monotone in each p_i it follows that $\hat{\theta}(y,n)$ is non-decreasing in each y_i and non-increasing in each n_i. In other words, if there were more failures than observed, or if there were the same number of failures in fewer trials, then $\hat{\theta}$ would be higher. We will call this the 'logical ordering' property. Intuitively, this is an essential logical property, not only of an estimator of θ but of any summary measure of failure rate. Some commonly used confidence limits do not possess the logical ordering property. For instance, approximate Wald-type limits do not, the main difficulty being that the variance estimates do not have the same property. Likelihood-based upper limits on the other hand do not involve an explicit variance estimate.

We are specifically interested in showing that $l_r(y,n)$ and $u_r(y,n)$ have the logical ordering property. There are several reasons why this is of interest. First, the logical ordering property is compelling in its own right. It is hard to imagine how more failures could be used to justify a lower assessment of failure risk. Second, Kabaila (2005) has shown that the computation of the Buehler limit is greatly simplified when the designated statistic has this property. In the same paper, it is demonstrated that the logical ordering property guarantees that Buehler bounds exist, whereas it is easy to construct examples where Buehler bounds do not exist by choosing a perverse designated statistic. Finally, since Buehler bounds based on the profile likelihood function have been recommended on the basis of their efficiency, the logical ordering property for this particular class of statistics becomes of interest.

2. Series system upper limits

For a series system with k components, system failure occurs when at least one component fails and so the failure probability is $\theta(p) = 1 - \prod_{i=1}^{k}(1-p_i)$ and the MLE is $\hat{\theta} = 1 - \prod_{i=1}^{k}(n_i - y_i)/n_i$. We want computational formulas for $l_r(y,n)$ and $u_r(y,n)$ and aim to show that these are non-decreasing in each y_i and non-increasing in each n_i. We first deal with some extreme cases.

Some $y_i = n_i$. The MLE $\hat{\theta} = 1$ if, and only if, some $y_i = n_i$. The upper limit u_r in this case also equals 1.0.

All $y_i = 0$. The MLE $\hat{\theta} = 0$ if, and only if, all $y_i = 0$. Let J be the set of indices for which $n_j = \min_{i=1}^{k} n_i := n_{\min}$. Then

$$L(p) \propto \prod_{i=1}^{k}(1-p_i)^{n_i} = (1-\theta)^{n_{\min}} \prod_{i \notin J}(1-p_i)^{n_i - n_{\min}}$$

which, subject to the restriction that $\theta = t$, has maximum value $L(t) = (1-t)^{n_{\min}}$ achieved when $p_i = 0$ for $i \notin J$ and the other p_i satisfy $\prod_{i \notin J}(1-p_i) = 1-t$.

Henceforth, we assume that $0 < \hat{\theta} < 1$. The numerator of $\mathscr{L}(t)$ is the likelihood $L(p)$ partially maximised with the restriction that $\theta(p) = t$. Standard calculus of variations arguments imply that we should maximise the objective function

$$\log L(p) - \psi(\theta - t) = \sum_{i=1}^{k} y_i \log p_i + x_i \log(1-p_i) - \psi\left(1 - \prod_{i=1}^{k}(1-p_i) - t\right)$$

where for convenience we denote by $x_i = n_i - y_i$ the number of tested components that do not fail. The derivative with respect to p_i is

$$\frac{\partial \ell}{p_i} = \frac{y_i - n_i p_i}{p_i(1-p_i)} - \frac{\psi(1-\theta)}{(1-p_i)}$$

which when equated to zero implies that $p_i = y_i/(n_i - \psi(1-\theta))$. Therefore,

$$\mathscr{L}(t) = \frac{\prod_{i=1}^{k}(y_i/(n_i - \psi(1-\theta)))^{y_i}((x_i - \psi(1-\theta))/(n_i - \psi(1-\theta)))^{x_i}}{\prod_{i=1}^{k}(y_i/n_i)^{y_i}(x_i/n_i)^{x_i}} = \prod_{i=1}^{k} \frac{(1-\psi(1-\theta)/x_i)^{x_i}}{(1-\psi(1-\theta)/n_i)^{n_i}}$$

and the multiplier ψ solves

$$1 - t = \prod_{i=1}^{k} \frac{x_i - \psi(1-\theta)}{n_i - \psi(1-\theta)}.$$

Replacing $\psi(1-\theta)$ by λ and t by θ, the profile likelihood function is given by

$$\mathscr{L}(\theta) = \prod_{i=1}^{k} \frac{(1-\lambda/x_i)^{x_i}}{(1-\lambda/n_i)^{n_i}} \tag{1}$$

where λ solves

$$1-\theta=\prod_{i=1}^{k}\frac{x_i-\lambda}{n_i-\lambda} \tag{2}$$

and the multiplier $\lambda \leqslant \min(x_i) := \lambda_{\max} > 0$. The last inequality follows from our assumption that not all $y_i < n_i$. The multiplier λ ranges from $-\infty$ to $\lambda_{\max}$. As λ ranges from $-\infty$ to 0 to $\lambda_{\max}$, the value of θ obtained from (2) increases from zero to $\hat{\theta}$ to unity. At the same time, likelihood (1) increases from zero to its maximum value of unity and then back down to zero. Note that in (1), $(1-\lambda/x)^x$ is interpreted as 1 when $x=0$.

3. Monotonicity of support limits

Support limits convert directly into approximate upper limits through the approximate χ_1^2 distribution of $-2\log\mathscr{L}(\theta)$, so an approximate $(1-\alpha)$ upper limit is just the support limit at level $r=\exp(-z_\alpha^2/2)$. This section demonstrates that upper support limits possess the required monotonicity properties, and therefore approximate upper limits based on the profile likelihood function and its asymptotic chi-squared distribution. The upper support limit is defined by

$$\log(1-u_r)=\sum_{i=1}^{k}\log(x_i-\lambda)-\log(n_i-\lambda) \tag{3}$$

where $\lambda > 0$ satisfies

$$\log\mathscr{L}(\lambda)=\sum_{i=1}^{k}x_i\log(1-\lambda/x_i)-n_i\log(1-\lambda/n_i)=\log(r). \tag{4}$$

We want to show that u_r is a non-increasing function of each x_j.

Case 1: $\lambda_{\max}=0$. This case occurs iff $\min(x_j)=0$ iff $\hat{\theta}=1$. Letting $J=\{j : x_j=0\}\neq\emptyset$, it follows that u_r is constant with respect to the $x_j : j\notin J$. Increasing any $x_j : j\in J$ leads to a solution $u_r<1$. So we conclude that $u_r(x,n)$ is non-increasing in each x_j.

Case 2: $\lambda_{\max}>0$. We consider x as a continuous positive variable and show, without loss of generality, that $\mathrm{d}u_r/\mathrm{d}x_1\leqslant 0$ and that, therefore, $\mathrm{d}u_r/\mathrm{d}x_j\leqslant 0$ for all j. Note that u_r varies with x_1 both directly through (3) and indirectly through λ which solves $\log\mathscr{L}(x,\lambda)=\log r$ given in (4). Differentiating (3) we have

$$\begin{aligned}\frac{\mathrm{d}\log(1-u_r)}{\mathrm{d}x_1}&=\sum_{i=1}^{k}\frac{\mathrm{d}x_i/\mathrm{d}x_1-\mathrm{d}\lambda/\mathrm{d}x_1}{x_i-\lambda}-\frac{\mathrm{d}n_i/\mathrm{d}x_1-\mathrm{d}\lambda/\mathrm{d}x_1}{n_i-\lambda}\\&=\frac{1}{x_1-\lambda}-\sum_{i=1}^{k}\frac{\mathrm{d}\lambda/\mathrm{d}x_1}{x_i-\lambda}+\sum_{i=1}^{k}\frac{\mathrm{d}\lambda/\mathrm{d}x_1}{n_i-\lambda}\\&=\frac{1}{x_1-\lambda}-\frac{\mathrm{d}\lambda}{\mathrm{d}x_1}\sum_{i=1}^{k}\frac{y_i}{(x_i-\lambda)(n_i-\lambda)}\end{aligned} \tag{5}$$

Differentiating $\log \mathscr{L}(\lambda, x)$ in (4) we have

$$\frac{\partial \log \mathscr{L}}{\partial \lambda} = -\lambda \sum_{i=1}^{k} \frac{y_i}{(x_i - \lambda)(n_i - \lambda)}, \quad \frac{\partial \log \mathscr{L}}{\partial x_1} = \log(1 - x_1/\lambda) + \frac{\lambda}{x_1 - \lambda}.$$

Since λ satisfies $\log \mathscr{L}(\lambda, x) = \log(r)$ it follows that

$$\frac{\partial \log \mathscr{L}}{\partial x_1} = -\frac{d\lambda}{dx_1} \frac{\partial \log \mathscr{L}}{\partial \lambda} = \lambda \frac{d\lambda}{dx_1} \sum_{i=1}^{k} \frac{y_i}{(x_i - \lambda)(n_i - \lambda)}$$

and so from (5)

$$\begin{aligned} \frac{d\log(1 - u_r)}{dx_1} &= \frac{1}{x_1 - \lambda} - \frac{1}{\lambda} \frac{\partial \log \mathscr{L}}{\partial x_1} \\ &= \frac{1}{x_1 - \lambda} - \frac{\log(1 - \lambda/x_1)}{\lambda} - \frac{1}{x_1 - \lambda} \\ &= -\frac{\log(1 - \lambda/x_1)}{\lambda}. \end{aligned}$$

It is easy to check that this expression is never negative for $\lambda \leqslant \lambda_{\max}$. For the upper limit u_r, the value of the multiplier is between zero and $\lambda_{\max}$ and we conclude that $d\log(1 - u_r)/dx_1 > 0$, which implies that $du_r/dx_1 < 0$ and that $du_r/dy_1 > 0$. For lower limit l_r, the value of λ happens to be negative but we again conclude that $dl_r/dy_1 > 0$. An almost identical series of calculations leads to the expression

$$d\log(1 - u_r)/dn_1 = -\log(1 - \lambda/n_1)/\lambda$$

which implies that $du_r/dn_1 < 0$ and $dl_r/dn_1 < 0$. Thus we have shown that $l_r(y, n)$ and $u_r(y, n)$ have the logical ordering property.

These results extend immediately to the parallel circuit where an upper/lower limit for the failure probability is a lower/upper limit for the reliability, i.e. probability of non-failure. The above derivation follows by reversing the roles of p and q. Showing the logical ordering property for a system of arbitrary design remains an open problem.

4. Illustration on forensic DNA matching

A standard problem in forensic DNA matching is to assess the probability of a suspect matching a particular DNA profile associated with the crime. Our example is taken from Chakraborty et al. (1993), who describe frequencies of a four-locus profile extracted from the FBI Caucasian database, displayed in Table 1. Under the common assumption that frequencies of the different loci are independent, the proportion θ of the population with the particular profile is the product of the four probabilities p_i of each locus. Replacing these four probabilities with their empirical estimates gives the estimate $\hat{\theta} = 8.888 \times 10^{-5}$. As noted by Balding (1995), the sampling distribution of this estimator is extremely skew and is highly likely to be smaller than the true probability θ. In a legal context, it is more pertinent to give an upper limit for θ with a high confidence coefficient.

Table 1
Frequencies for the four-locus profile, from Chakraborty et al. (1993)

	Frequency	Number	Proportion
Locus 1	35	1190	0.0294
Locus 2	169	1584	0.1067
Locus 3	128	1188	0.1077
Locus 4	408	1552	0.2629

Table 2
Effect of changing counts by ±1 on lower 99% limit, maximum likelihood estimate and 99% upper limit (all multiplied by 100,000)

y_1	y_2	y_3	y_4	Lower	$\hat{\theta}$	Upper
36^+	169	128	408	5.591	9.142	14.381
35	170^+	128	408	5.441	8.941	14.118
35	169	129^+	408	5.452	8.958	14.142
35	169	128	409^+	5.422	8.910	14.072
35	169	128	408	5.408	8.888	14.038
34^-	169	128	408	5.225	8.634	13.694
35	168^-	128	408	5.375	8.836	13.958
35	169	127^-	408	5.364	8.819	13.933
35	169	128	407^-	5.395	8.866	14.004

The probability $\theta = p_1p_2p_3p_4$ of the four-locus profile is identical to the probability of failure in a parallel circuit with component failure probabilities p_i. There is an obvious and close connection between parallel and series circuits. The probability of failure θ in the series circuit is one minus the probability of failure ϕ in the complementary parallel circuit with failure probabilities $q_i = 1 - p_i$. So an upper limit for the failure probability θ in the parallel circuit is equal to one minus a lower limit for probability of failure ϕ in the series circuit.

For the given data $y = (35, 169, 128, 408)$ and $n = (1190, 1584, 1188, 1552)$, we obtained the 99% lower limit 0.99985962 for ϕ and therefore an upper limit 14.038×10^{-5} for θ. Table 2 gives both upper and lower limits for θ (though only upper limits would be of interest in forensic DNA profiling) when each of the counts in y are increased by 1 (indicated by "+") and decreased by 1 (indicated by "−"). Both upper and lower limits satisfy the monotonicity property—since the statistics in the upper/lower sections of the table are higher/lower than the statistics for the observed data set.

5. Conclusion

It has been shown that profile likelihood support-based upper and lower limits for the failure rate of a series or parallel circuit are monotonically non-decreasing functions of the estimated component failure rates. This is of importance in constructing efficient and stable algorithms for calculating Buehler bounds for failures rates of such systems. The main results can be easily generalised to products of multinomial rather than binomial probabilities (available from the author).

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