A Constrained Coalitional Approach to Price Formation*

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Abstract

Since Edgeworth (1925) it has been understood that the Bertrand price setting game does not produce a pure strategy equilibrium in a number of simple settings. Two prominent examples are price competition with convex costs and spatial competition with finite buyers.

Building on work by Hamilton MacLeod and Thisse (1991), this paper develops an alternative model of price formation. We examine as a non-transferable utility coalitional game the set of outcomes that are feasible in the Bertrand price setting game. We prove that in spatial models with finite buyers the core of this NTU coalitional game is equivalent to the set of outcomes that can be produced by undercut-proof prices. Moreover we show that in the presence of convex costs the market clearing price is always in the core, and that where competition exists on both sides of the market there is a sense in which the core collapses to only admit market clearing outcomes.

In some settings the constrained coalitional price setting game produces results that contradict the predictions of the Bertrand price setting game. Where this occurs core outcomes of the constrained coalitional price setting game tend to be more efficient than the corresponding Bertrand-Nash equilibrium. Moreover, we show that in vertically related markets double-marginalisation is never a core outcome.

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Within industrial organisation price formation tends to be modelled as a Bertrand price setting game.[1] Implicit in the Bertrand price setting game is the assumption that each seller is constrained such that she must offer the same unit price to every potential buyer. This assumption effectively constrains the range of payoffs that may arise as outcomes of the game.

The Bertrand price setting game has proved to be problematic in a number of settings. In numerous simple models the Bertrand price setting game fails to produce a pure strategy Nash equilibrium. Some notable examples include competition in a market for a homogeneous good in which sellers face non-trivial capacity constraints or upward sloping marginal cost curves, and spatial competition with finite buyers. Edgeworth (1925) was among the first to identify the absence of a pure strategy equilibrium stating that, in the presence of capacity constraints,

> "...there will be an indeterminate tract through which the index of value oscillate, or rather will vibrate irregularly for an indefinite length of time. There will never be reached that determinate position of equilibrium..."

Edgeworth, writing before the concept of a mixed strategy had been developed, proposed that capacity constraints would instead give rise to the price cycle that we now know as the Edgeworth cycle.[2]

When faced with a model for which a pure strategy Bertrand-Nash equilibrium does not exist, authors have adopted a number of different approaches. Those wishing to retain a one-shot price setting game are forced to seek out the mixed strategy equilibria,[3] while authors willing to redefine the structure of the price setting game have employed repeated games with either simultaneous or sequential moves.[4] Neither approach has proven to

[1]Prices can also be backed out of the Cournot quantity setting game. However, as Kreps and Scheinkman (1983) have demonstrated, Cournot outcomes will be produced by a Bertrand price setting game if sellers can unilaterally commit to a capacity in advance of the market.

[2]The original description of the price cycle can be found in Edgeworth (1925). Maskin and Tirole (1988) show how an Edgeworth can arise as a Markov Perfect Bertrand-Nash equilibrium of a infinitely repeated sequential moves price setting game.

[3]See, for example, Shilony (1977).

[4]A notable example of the latter approach is Eaton and Engers (1990).

be particularly satisfactory. Mixed strategy equilibria can be difficult to interpret when the prices being modelled will persist for more than an instant, while dynamic games frequently lack predictive power due to the large range of prices that may be supported as an equilibrium.

An obvious alternative to the Bertrand price setting game is to model transactions as a transferable utility (TU) coalitional game. TU games have the advantage of being relatively easy to work with, however TU coalitional games do not place any restriction on the ways in which the value created by a coalition may be allocated between the members of that coalition. In the context of Industrial Organisation (IO) this can be problematic. Many IO models assume that for either technological or regulatory reasons sellers are constrained such that they must each set a single linear price for their product. Absent the possibility of side payments, the range of outcomes that are feasible in a market in which a linear price constraint operates, is much smaller than is the case in a TU coalitional game.

If we accept that, for a given market, linear prices are not simply the outcome of the price setting game, but rather the consequence of a constraint exogenously imposed upon transactions, the TU approach is not acceptable. However, there does exist a third alternative, a synthesis of Bertrand price setting and a coalitional game, that obeys the linear price constraint while still delivering the precise and robust results that we seek. This game, which we term the *constrained coalition price setting game*, examines as a non-transferable utility (NTU) coalitional game the set of outcomes that are feasible in the Bertrand price setting game.

This approach is not without precedent. Guesnerie and Oddou (1981) analyse as an NTU coalitional game, second-best taxation to create a public good. In their analysis an allocation of value is only feasible in the economy if it can be implemented by a linear tax on private wealth. Similarly, Spulber (1986) employs an NTU framework to determine where a multi-product natural monopoly, constrained to employ a linear price schedule for its products, will be robust against an entrant targeting a specific subset of its customers.

Probably the most complete existing treatment of this game can be found in Hamilton, MacLeod & Thisse's (1991) solution to the location choice problem on the Hotelling line with linear transport cost. As D'Aspremont, Gabszewicz and Thisse (1979) have shown, there does not exist a pure strategy Bertrand-Nash equilibrium to the price setting game when the two sellers are closely spaced (but not collocated) on the line. In order to extract a pure strategy solution, Hamilton et. al. (1991) consider as an NTU coalitional game those allocations that can be implemented by a price pair. In Hamilton et. al.'s price setting game sellers may only select a price pair if it implements an outcome within the core of this NTU game.[5]

This paper develops the formalism necessary to apply the constrained coalitional price setting game to markets for a homogeneous, or spatially differentiated good.[6] Armed with this technique we proceed to tackle a series of previously intractable and anomalous problems in IO. The game is shown to be very flexible, making no assumptions concerning the distribution of "bargaining power" between buyers and sellers.[7] Consequently, the constrained coalitional price setting game may be employed in the analysis of intermediate markets where it is possible for firms to exercise both monopoly and monopsony power.

The approach taken in this paper was inspired by Telser's (1987) study of efficient cooperation and competition. Telser characterises games in which each player acts unilaterally — such as Bertrand price setting — as being competitive, while games in which decisions are made via coalitional interaction are termed cooperative. Telser conjectures that in order for a market to operate efficiently it requires a balance of co-operative and competitive forces. For example, fierce competition can be socially harmful when it leads to wasteful duplication, whereas a degree of co-operation in activities such

[5]Intuitively, the ability of sellers to dictate which core outcome will be implemented can be seen as analogous to the ability of sellers to commit to take-it or leave-it offers in the Bertrand price setting game.

[6]Hamilton et. al.'s (1991) treatment of the Hotelling line with linear transport costs is a special case of the latter.

[7]The terms buyer and seller are used in place of the more traditional consumer and firm in recognition of the fact that firms may be either a buyer or a seller in the market, and in order to avoid the common characterisation of consumers as passive participants in the game. Throughout the paper we will refer to buyers with masculine pronouns and sellers with feminine pronouns. This convention is adopted in the interests expositional simplicity.

as research and development has the capacity to enhance efficiency.

An important distinction between Telser's (1987) characterisation and that adopted here is that Telser views core outcomes as being the product of a coopperative interaction between players. This characterisation fails to account for the fierce competition that may arise within a coalitional game. In contrast we view the a coalitional game as containing a balance of competitive and cooperative forces with players cooperating to create value before competing over the division of this value.[8]

The structure of this paper is as follows: Section 1 sets out the formalism of the constrained coalitional price setting game. Whereas Hamilton et. al. (1991) consider only competition between two sellers on the Hotelling line, the methodology developed in this paper can be applied to quite general class of market in which homogeneous, or spatially differentiated goods are produced by sellers, and sold to buyers. Included in this section is a method for endogenising the rationing of output where the capacity of sellers at a given price is less than the demand of buyers. We also develop an important result that relates the cores of a TU and constrained coalitional analysis of a market.

For the special case of the Hotelling line, Hamilton et. al. (1991) have shown that an allocation lies in the core of the constrained coalitional price setting game if and only if it can be implemented by *undercut-proof* prices. A price vector is undercut-proof if no seller can gain by unilaterally reducing her price. In section 2 we show that this undercut-proofness result extends, with a minor caveat, to a broad class of spatial model.

Section 3 considers the general case of a market for a homogeneous good in which sellers face convex cost functions. As Edgeworth (1925) demonstrated, when faced with convex costs the market may fail to possess a pure strategy Bertrand Nash equilibrium. It is shown that the market clearing price always implements an allocation in the core of the constrained coalitional price setting game for this model, and that where demand becomes

[8]See MacDonald and Ryall (2004) for an insightful characterisation of the role that competition plays in appropriation of value in a TU coalitional game. Similar factors are at work in an NTU coalitional game, although here the inability of players to arbitrarily assign value between the members of a coalition complicates the analysis.

“fine” and there is competition between sellers, there is a sense in which the core collapses to only admit allocations corresponding to market clearing prices. We also demonstrate that the linear price constraint may lead to inefficient outcomes lying within the core.

The final case considered in this paper concerns vertical related markets where an input sold in an upstream market is transformed into a product sold in a downstream market. Once again it is shown that it is always a core outcome for both markets to operate at the market clearing price. Moreover, the double-marginalisation that may arise in a Bertrand analysis of this market is shown to be an non-core outcome in the constrained coalitional price setting game.

The paper concludes with a discussion of the model and results, including the implications for anti-trust policy.

1 The Price Setting Game

In this section we formalise the structure of the constrained coalitional price setting game for application to exchange between distinct sets of buyers and sellers in markets for homogeneous or spatially differentiated goods. As is the case with all NTU coalitional games, this game is completely described by the set of players N,[9] and a correspondence V from coalitions to subsets of $\mathbb{R}^{|N|}$, that indicates the set of payoffs that are feasible for each given coalition of players.

The first step in constructing V is to set out each player's payoff as a function of the primitives of the model — buyer demand functions and seller cost functions — as they depend on the parameters of price and quantities traded. Next, we determine the ways in which the output produced by sellers may be rationed between buyers given the prevailing prices and active coalition. As is the case in the Bertrand price setting game rationing is not trivial. This can best be seen in the presence of capacity constraints where the low priced seller is unable (or unwilling) to satisfy market demand, as well

[9] In the language of coalitional game theory N is the *grand coalition* consisting of all the players — buyers and sellers — active in the market.

as where sellers "tie" in the terms of trade that they offer to a given buyer.[10] Our approach to rationing is axiomatic. We require rationing to be consistent with the principal that all players in the market seek to unilaterally optimise their trades given the prevailing prices. Where more than one rationing scheme satisfies this requirement we allow the players to choose between the alternatives. The section concludes with a result linking the cores of TU and NTU coalitional games, and a guide to interpreting the core.

Formally, given finite sets B of buyers, and S of sellers, let $p_j \in \mathbb{R}$ denote the price set by the seller $s_j \in S$, and let $q_{ij} \in \mathbb{R}_+$ denote the quantity purchased by the buyer $b_i \in B$ from the seller s_j. Given prevailing prices $p = \{p_j\}_{j \in S}$ and quantities traded $q = \{q_{ij}\}_{i \in B, j \in S}$, the payoff to a player k is denoted by the real valued function $x_k(p, q)$.

Given that this is a coalitional analysis it is important to understand the payoffs that players receive when a coalition $G \subseteq N = B \cup S$ of players deviate and trades amongst themselves. Rather than make $x = \{x_k\}_{k \in N}$ a function of G directly, we adopt the assumption that trades are not possible across the boundary of G. Formally $q_{ij} = 0$ where $\{i, j\} \cap G \neq \varnothing$ and $\{i, j\} \nsubseteq G$. We can interpret this assumption as stating that when a coalition G deviates, the members of the coalition leave "the market" and trade amongst themselves.

Throughout this paper it is assumed that the payoff to a seller $s_j \in S$ takes the form,

$$x_{s_j}(p, q) = p_j \sum_{i \in B} q_{ij} - C_j\Big(\sum_{i \in B} q_{ij}\Big), \tag{1.1}$$

where the function $C_j(\cdot)$ denotes the cost to s_j of producing the quantity $\sum_{i \in B} q_{ij}$. We assume that cost functions are convex such that $C_j' \geq 0$ and $C_j'' \geq 0$ for all $s_j \in S$. In the extreme case where a seller s_j is subject to the capacity constraint $\bar{q}$, we write $C(\vartheta) = \infty$ for all $\vartheta > \bar{q}$.

Similarly, each buyer $b_i \in B$ receives the payoff,

$$x_i(p, q) = \int_0^{\sum_{j \in S} q_{ij}} D_i^{-1}(\vartheta) d\vartheta - \xi_i(p, q_i), \tag{1.2}$$

where the function $D_i^{-1}(\cdot)$ represents buyer i's inverse demand and ξ_i represents the financial cost to buyer i of purchasing the quantities $q_i = \{q_{ij}\}_{j \in S}$

[10] Thus rationing in this model also functions to endogenise the "tie breaking rule".

at the prices p. In sections 3 and 4 where the product being traded is homogeneous ξ_i takes the form $\xi_i(p, q_i) = p \cdot q_i$, while in section 2 where products are spatially differentiated ξ_i also contains a term representing the transport costs incurred by the buyer. The only requirements that we place on a buyer's inverse demand at this stage is that D_i^{-1} is integrable and $dD_i^{-1}(\vartheta)/d\vartheta \leq 0$.

1.1 Rationing

In the Bertrand price setting game q is stated as a function of p.[11] The relationship between p and q in the Bertrand price setting game is consistent with all players in the market — buyers and sellers — unilaterally optimising their trades given the prevailing price vector. However, in most models there exists at least one price vector p for which more than one q is consistent with this unilateral optimisation. In the Bertrand price setting game this ambiguity is resolved through the use of a "rationing rule".[12]

Consider, for example, the case of Bertrand competition with homogeneous goods and capacity constraints examined by Kreps and Scheinkman (1983). In this model one seller may designate a price that lies below that of its rival, without being able (or willing) to satisfy market demand at that price. Kreps and Scheinkman employ the *efficient rationing* rule. Formally, q satisfies efficient rationing if and only if it maximises the aggregate surplus that accrues to buyers given the prevailing prices and the willingness of sellers to supply at these prices. Efficient rationing is not uncontroversial as it precludes sellers employing rules such as first-come first-served, and may require a seller to have detailed knowledge of each buyer's individual preferences. Moreover, even where sellers are able to institute efficient rationing it will not generally be in the best interests of sellers as efficient rationing tends to minimise the total quantity traded. These features are illustrated in example 1.1 below.

Hamilton et. al. (1991) do not consider how to incorporate rationing into

[11] In this way the price vector uniquely determines the vector of payoffs, and the problem is reduced such that seller prices become the only strategic variables.

[12] See Tirole (1988, pp. 212–215) for a discussion of the role of rationing in the Bertrand price setting game and examples of rationing rules.

their price setting game as rationing in the Hotelling model is straightforward. However, for the range of cases considered in this paper the issue of rationing must be addressed. Our approach is axiomatic. We wish to restrict trades such that they are consistent with players unilaterally optimising their trades given the prevailing price vector and active coalition.

Defining a unilateral optimum where the strategic variables — the q_{ij}'s — are shared by two players, is not a straightforward matter. Given the context we permit a player to unilaterally reduce any component of q to which he is a party. At the same time we require that a player gains consent from his trading partner if he wishes to increase the quantity to be traded. These two conditions ensure that no buyer can be forced to purchase a quantity that exceeds his demand, and no seller can be forced to sell a quantity that is greater than her willingness to supply, in q or a unilateral deviation that would exclude q. As the strategic environment is one of unilateral action we permit a player to coordinate a variation across his vector of trades.

Definition 1. The trades $q = \{q_{ij}\}_{i \in B, j \in S}$ constitute a *rational rationing scheme* for price vector p and coalition G if and only if for all $i \in N$, and $\hat{q}_i \in \mathbb{R}^{|S|}$ if i is a buyer or $\hat{q}_i \in \mathbb{R}^{|B|}$ if i is a seller, such that,

i. $\hat{q}_{ij} = 0$ where $\{i, j\} \cap G \neq \varnothing$ and $\{i, j\} \nsubseteq G$, and,

ii. $x_i\big(p, (q_{-i}, \hat{q}_i)\big) > x_i(p, q)$,

there exists at least one $j \in N$ such that $\hat{q}_{ij} > q_{ij}$ and $x_j\big(p, (q_{-i}, \hat{q}_i)\big) < x_j(p, q)$.

If q satisfies the efficient rationing rule then q is a rational rationing scheme, however the converse is not necessarily true. To see this note that efficient rationing assumes that neither buyers nor sellers exceed their respective demands or willingness to supply at the prevailing price. Moreover, given that under the efficient rationing rule buyers are maximising their collective payoffs subject to the willingness to supply of sellers, any unilateral deviation that benefits the deviating player must require at least one trading partner to increase the quantity that they exchange, violating either a buyer's demand, or seller's willingness to supply. The following example illustrates the relationship between efficient rationing, rational rationing, and trades that do not satisfy either rationing rule.

Example 1.1. Consider a market in which $B = \{b_1, b_2, b_3\}$ and $S = \{s_1, s_2\}$. Suppose that both sellers produce identical products at a constant marginal cost of zero and that each seller has a capacity of two. Each buyer demands one unit, with buyers 1 and 2 willing to pay a price of 2 for the product while buyer 3 has a willingness to pay of 1.

Suppose that the prices produced by the first stage game are $p_1 = 1/2$ and $p_2 = 3/2$, and consider the following trades:

i. Buyers 1 and 2 each purchase a unit from seller 1 and buyer 3 does not trade.

ii. Buyers 1 and 3 each purchase a unit from seller 1 and buyer 2 purchases a unit from seller 2.

iii. Buyers 1 and 2 each purchase a unit from seller 2 and buyer 3 purchases a unit from seller 1.

The first set of trades satisfies both efficient rationing and rational rationing. The buyers with the highest valuation for the product purchase from the seller with the lowest price thereby maximising the aggregate surplus that accrues to buyers, creating an aggregate surplus of 3 for the buyers and 1 for the sellers. Buyer 3 does not receive any of the good and cannot profitably exploit seller 2's excess capacity at the price $p_2 = 3/2$ as this price is greater than buyer 3's valuation for the product.

The second set of trades sees buyer 3 trading with the low price seller while buyer 2 is left to trade with the high price seller. Such a situation might arise if buyers 1 and 3 arrive at the store before buyer 2, exhausting the capacity of seller 1. By the time buyer 2 arrives at the store he has no choice but to purchase the good from seller 2 at the higher price. In this case the aggregate surplus that accrues to buyers has fallen to 5/2 while the total seller payoffs have risen to 5/2. This second set of trades clearly do not satisfy efficient rationing as they do not maximise the aggregate buyer surplus. The trades remain rational however, as no buyer can profitable exploit the excess capacity of seller 2 and the demand of every buyer is satisfied.

The final set of trades do not satisfy either rationing rule. The total surplus that accrues to buyers is 3/2; half of the maximum. Moreover, given that seller 1 has one unit of excess capacity, buyer 1 could reduce the quantity that he purchases from seller 2 to zero, instead purchasing one unit of the good from seller 1. This variation strictly improves buyer 1's payoff without reduce seller 1's payoff, violating definition 1. ■

In the constrained coalitional price setting game players collectively determine the pair (p, q). However, while players may select any $p \in \mathbb{R}^{|S|}$, we constrain the choice of q such that it satisfies rational rationing. Henceforth we write,

$$Q^r(p, G) = \left\{q \in \mathbb{R}_+^{|B \times S|} : q \text{ is a rational rationing scheme for } (p, G)\right\}, \quad (1.3)$$

to indicate the set of trades that satisfy rational rationing for a given price vector p and coalition G. Intuitively, while the players are committing to a set of trades at the same time that they are negotiating prices, they require q to be robust against unilateral deviations once the price is set.

1.2 Solving for Prices

At the centre of the constrained coalitional game is the set of allocations that can be feasibly generated by players. For a given coalition $G \subseteq N$, the set of feasible allocations is defined as,

$$V(G) = \Big\{x_G\big((p_G, p_{-G}), q\big) : p_G \in \mathbb{R}^{|S \cap G|}, q \in Q^r\big((p_G, p_{-G}), G\big)\Big\}, \quad (1.4)$$

where $x_G = \{x_k\}_{k \in G}$ and $p_G = \{p_j\}_{j \in S \cap G}$. Thus every allocation in the feasible set must be able to be implemented by a vector of linear prices p and the trades q. Moreover, the quantities traded are robust to unilateral deviations given prevailing prices and the active coalition. Given the assumption that trades cannot occur across the boundary of the active coalition, $V(G)$ is insensitive to variations in p_{-G} and as such p_{-G} may be assigned an arbitrary value.

$V(N)$ is closely related to the set of outcomes that may be produced by the Bertrand price setting game, differing only insofar as $V(N)$ admits multiple outcomes for a given p where $Q^r(p, N)$ is not a singleton. The constrained coalitional approach differs from Bertrand in the way in which a solution is extracted from $V(N)$. For the purposes of this model the core of an NTU coalitional game (N, V) is defined as follows:

Definition 2. An allocation x is in the core of (N, V) if and only if for all $G \subseteq N$ and $y \in \mathbb{R}^{|G|}$, if $y_k \geq x_k$ for all $k \in G$, with strict inequality for one k, then $y \notin V(G)$.

This definition differs from that of Myerson (1991) in one important respect. Myerson only allows a coalition G to block an allocation x if there exists an allocation $y \in V(G)$ such that $y_k > x_k$ for all $k \in G$. In contrast to the definition of the core of a transferable utility type coalitional (TU) game, the choice of strength for the inequalities is not innocuous here.[13] As example 1.2 illustrates, weak inequalities create a more stringent condition that is harder to satisfy than strict inequalities, giving us greater confidence in the robustness of allocations in the core to re-contracting. Intuitively the use of weak inequalities can be seen as implying a weak preference for altruism.

Example 1.2. The difference between the two cores can be seen clearly in the two player NTU game (N, V) where $N = \{1, 2\}$, $V(1) = V(2) = \{0\}$ and $V(N) = \big\{(x_1, x_2) : \max\{x_1, x_2\} \leq 1\big\}$. The set of allocations that are feasible for the grand coalition are illustrated in figure 1 as the area $ABC0$. By Myerson's definition the core of (N, V) is the border ABC, whereas according to definition 2 the core of (N, V) is the point B. Invoking weak inequalities lowers the dimension of the core and increases the precision of the model. ■

The constrained coalitional price setting game belongs to a special class of NTU coalitional that can be generated by applying a constraint to a TU

[13] In a TU game weak and strict inequalities are equivalent. If one member of a coalition in a TU game strictly benefits from a deviation that leaves the other members of the coalition no worse off, some portion of that benefit can be redistributed between the remaining players of the coalition to create a strict benefit for every player. In an NTU game such an arbitrary re-allocation may not be possible.

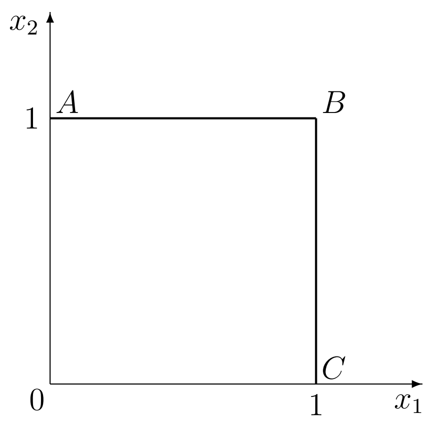

Figure 1: Comparing NTU Cores

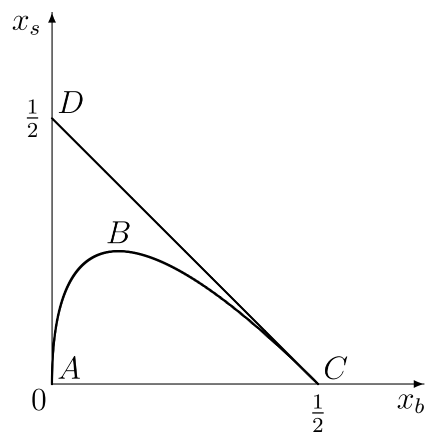

Figure 2: Bilateral Monopoly

coalitional game. Consider a real valued function $w : 2^N \longrightarrow \mathbb{R}$ satisfying,

$$V(G) \subseteq \Big\{x_G : \sum_{k \in G} x_k \leq w(G)\Big\}, \quad \forall G \subseteq N, \tag{1.5}$$

and define a correspondence W such that,

$$W(G) = \Big\{x_G : \sum_{k \in G} x_k \leq w(G)\Big\}, \quad \forall G \subseteq N. \tag{1.6}$$

(N, W) is the NTU representation of the TU coalitional game (N, w). There is a sense in which the game (N, V) can be generated by applying a constraint to the game (N, W), hence the use of the term *constrained* in the name of the price setting game.[14] The natural candidate for $w(G)$ in the price setting game is the maximum value that can be created by trades between the members of coalition G.

Constrained coalitional games can be easier to work with than other NTU games thanks to the following useful result:

Lemma 1. *Suppose that* (N, V) *and* (N, w) *satisfy* (1.5)*,*

$$\text{Core}(N, w) \cap V(N) \subseteq \text{Core}(N, V). \tag{1.7}$$

[14]Not all NTU coalitional games can be represented as the product of such a constraint. Consider the two player NTU game in which $V(1, 2) = \{x : x_1 x_2 \leq 1\}$. There does not exist a real valued function w satisfying (1.5) as the sum $x_1 + x_2$ is unbounded.

Proof. The intersection in (1.7) is required as an allocation must be feasible for the grand coalition in (N, V) in order to lie in the core of (N, V). Now, consider an allocation $x \in \text{Core}(N, w) \cap V(N)$. It must be the case that,

$$\sum_{k \in G} x_k \geq w(G), \quad \forall G \subseteq N. \tag{1.8}$$

From the construction of $W(G)$ it follows that there does not exist a $G \subseteq N$ and $y \in W(G)$ such that $y_k \geq x_k$ for all $k \in G$, with strict inequality for one k. Moreover, given that $V(G) \subseteq W(G)$, no such y exists in $V(G)$ either. □

Lemma 1 states that if an allocation lies in the core of a TU analysis of a market, and is feasible in the sense that it can be implemented via a linear price mechanism, that allocation will also lie in the core of the constrained coalitional price setting game. Moreover, given that all allocations in the core of a TU game are efficient, if $\text{Core}(N, w) \cap V(N) \neq \varnothing$, there exists an efficient allocation in $\text{Core}(N, V)$.

Lemma 1 is not generally sufficient to completely define the core of a constrained coalitional game. Some allocations in $\text{Core}(N, w)$ may not be feasible in (N, V). Moreover, as a consequence of constraining (N, V), an allocation y that lies outside of $\text{Core}(N, w)$ may still be found within $\text{Core}(N, V)$ if sufficient allocations have been removed from V to prevent y from being blocked. Nevertheless, lemma 1 is particularly useful for proving existence, and provides an important link between the outcomes of unconstrained bargaining and the constrained coalitional price setting game.

The following example illustrates the construction of a feasible set, and the implications of lemma 1, for a bilateral monopoly market.

Example 1.3. A seller s with unlimited capacity and a constant marginal cost of zero, trades a quantity q to a buyer b. The buyer's payoff is,

$$x_b(p, q) = q - \frac{q^2}{2} - pq, \tag{1.9}$$

where p is the market price, while the seller's payoff is $x_s(p, q) = pq$.

Given this payoff structure rational rationing admits a unique quantity for every choice of price. In this example we have but one buyer and one seller — precluding competitive effects on either side of the market — while the seller has unlimited capacity. It follows that the only constraint on the quantity traded is the buyer's demand at the prevailing price, specifically $Q^r(p, N) = \{1 - p\}$ for all $p \in [0, 1]$ and $q = 0$ otherwise. Substituting for these values we find that $x_b = (p^2 - 2p + 1)/2$ and $x_s = p - p^2$ when the players agree to some price $p \in [0, 1]$.

We are now in a position to generate the feasible sets. Substituting for p we find that $V(b) = V(s) = \{0\}$, and $V(N) = \{x : x_s = \sqrt{2x_b} - 2x_b\}$. The set $V(N)$ is illustrated by the curve ABC in figure 2. The greatest total surplus that can be created in this market is $\frac{1}{2}$, however this outcome can only be achieved when both the price and the seller's payoff are zero. In order for the monopolist to appropriate value in this example, she must destroy some of this total surplus. This feature of monopoly behaviour survives in the constrained coalitional price setting game because the seller must charge the same unit price for every unit sold, and the seller is unable to force the buyer to purchase a sub-optimally large quantity at that price.

It is straightforward to see that a TU coalitional game (N, w), with $w(b) = w(s) = 0$ and $w(N) = \frac{1}{2}$, satisfies the relationship set out in (1.5). The set $W(N) = \{x : x_b + x_s \leq \frac{1}{2}\}$ is illustrated in figure 2 as the area lying on and below the line CD.

In a two player NTU coalitional game of this type we can find the core by inspection. An allocation x is in the core of (N, V) if and only if $x \geq 0$ and there does not exist $y \in V(N)$ such that $y > x$. Applying this criteria to figure 2 we see that the core of (N, V) is the curve BC, while the core of (N, w) is the line CD. In accordance with lemma 1, the only point in Core(N, w) that is feasible in the game (N, V) — the point C — also lies in the core of (N, V). No other point in Core(N, w) can be achieved by the players in the game (N, V), illustrating the importance of the feasibility condition in lemma 1. At the same time the remaining allocations on the curve BC are only in Core(N, V) because the constraints that reduce the set $W(N)$ to the set $V(N)$ have removed all the allocation in $W(N)$ capable of blocking them.

Two features of Core(N, V) are a product of the NTU structure of this game, and cannot arise in a TU coalitional game. First, all but one point in Core(N, V) are inefficient. This is clearly the case as the set of payoffs that sum to the maximum available surplus is illustrated by the line CD. Second, and as a purely technical matter, Core(N, V) is not convex. ■

1.3 Interpreting the Core

The core of a constrained coalitional game can be very large — as is the case in the switching costs example of the next section — or may converge to a unique allocation as is the case in the convex costs model of section 3. Where the core encompasses a wide range of allocations we are left with the question: How can we imbue the core with predictive power?

In their examination of location choice on the Hotelling line Hamilton et. al. (1991) consider only the core allocation that delivers the highest payoff to sellers. Intuitively, this allocation can be viewed as the core allocation that arises when the sellers possess all the "bargaining power". Given that the Bertrand price setting game grants sellers the power to make final take-it or leave-it offers, an allocation that is preferred by all sellers can be regarded as the analogue of the Bertrand Nash equilibrium.

However, the remainder of the core has value as well. Such is the flexibility of the constrained coalitional price setting game that it can be applied to markets — such as intermediate markets — in which buyers are considered to possess some degree of bargaining power. The example presented in section 3 shows the effect that both monopoly and monopsony power have on the shape of the core. Exactly where in this core we would expect the market to operate will depend upon the "relative bargaining power" of the two sides.

2 Spatial Competition

Hamilton et. al. (1991) developed the constrained coalitional methodology in order to solve the problem of location choice on the Hotelling line. They adopted this technique because when firms are closely spaced on the line the Hotelling model does not possess a pure strategy Bertrand Nash equilibrium. In this section we extend Hamilton et. al.'s model to a general spatial setting with finite buyers, in the process proving that the central result developed by Hamilton et. al. for the Hotelling line remains valid in this setting. Specifically, it is shown that the core of the spatial market is equivalent to the set of payoffs that can be generated by *undercut-proof* prices. This fact allows us to reduce the often complex problem of finding the core to a single condition on each seller's price.

The essence of spatial competition can be captured in the form of each player's payoff. Each seller s_j is assumed to have unlimited capacity and face a constant marginal cost c_j such that,

$$x_j(p, q) = (p_j - c_j) \sum_{i \in B} q_{ij}. \tag{2.1}$$

While the payoff of each buyer b_i can be expressed as,

$$x_i(p, q) = \int_0^{\sum_{j \in S} q_{ij}} D_i^{-1}(\vartheta) d\vartheta - \sum_{j \in S} \big(\mathbb{1}_{q_{ij} > 0} K_{ij} + q_{ij}(p_j + k_{ij}) \big). \tag{2.2}$$

Here $K_{ij} \in \mathbb{R}_+$ is defined as the fixed cost that b_i faces when trading with s_j, $\mathbb{1}_{q_{ij} > 0}$ is an indicator function that takes the value 1 if $q_{ij} > 0$ and 0 otherwise, and $k_{ij} \in \mathbb{R}$ is the per unit transport cost.

While the values of K_{ij} and k_{ij} can be selected to satisfy a particular geometric structure, this need not be the case. The form of buyer payoffs set out in (2.2) admits the possibility of either buyers or sellers occupying multiple locations.

Throughout this section we use the "switching costs model" to illustrate step-by-step the way in which a spatial problem can be solved through the use of the constrained coalitional price setting game. In its simplest form the switching costs model consists of two locations with a buyer and seller collocated at each. Each buyer has unit demand and must decide between buying locally or incurring transport cost $t > 0$ to purchase from the seller at the other location.

Several authors have tackled the switching costs model and the model has spawned a number of innovative solutions. The mixed strategy Bertrand Nash equilibrium to this model was characterised by Shilony (1977), Eaton & Engers (1990) found Markov perfect equilibria of an infinitely repeated sequential moves Bertrand price setting game, while Shy (2001) proposed that an *undercut-proof* criteria is the appropriate solution concept to apply to the model. While it turns out that Shy's solution (almost) coincides with the solution proposed here, Shy does not provide a theoretical justification undercut-proofness as an equilibrium concept. Rather, Shy justifies the undercut-proof criteria on the grounds of existence and tractability.

In the first part of this example we show how the switching costs model can be parameterised in terms of the payoff functions set out in (2.1) and (2.2), and demonstrate that the model does not posses a pure strategy Bertrand Nash equilibrium.

Example 2.1. In the switching costs model we have $B = \{b_1, b_2\}$ and $S = \{s_1, s_2\}$, with $K_{ij} = t$ for $i \neq j$ and $K_{ij} = 0$ otherwise. To complete the characterisation we have $c_1 = c_2 = c$, $k_{ij} = 0$ for all i and j, and,

$$D_i^{-1}\Big(\sum_{j \in S} q_{ij}\Big) = \begin{cases} v & \sum_{j \in S} q_{ij} \in [0,1] \\ 0 & \sum_{j \in S} q_{ij} \in (1,\infty) \end{cases} \quad \forall i \in B, \tag{2.3}$$

where $v > 2t + c$ is the valuation that the buyers place on a unit of output.

One can readily check that this model does not possess a pure strategy Bertrand Nash equilibrium. It is straightforward to see that we must have $p_i \geq c$ for all $i \in \{1, 2\}$ in a Bertrand Nash equilibrium. We are left with three possible cases:

Where $p_i > p_j + t$, seller s_i receives a payoff of zero as the buyer b_i can get a better deal by trading with seller s_j. Given that s_i can increase her payoff to $p_j + t - c > 0$ by matching s_j's price we cannot have $p_i > p_j + t$ in a pure strategy Bertrand Nash equilibrium.

Regardless of the tie breaking rule, for $p_i = p_j + t$ we must have either s_i or s_j trading less than one unit to b_i. It follows that at least one seller must be able to strictly increase her payoff by lowering her price by sufficiently small $\varepsilon > 0$; this excludes $p_i = p_j + t$ as a possibility.

Finally, for $p_i \in (p_j - t, p_j + t)$ note that s_i trades only with b_i and receives the payoff $x_{s_i} = p_i - c$. Given that this interval of prices is open, s_i can always find another price that is greater than p_i, but still lies within the interval $(p_j - t, p_j + t)$, thereby increasing her payoff. It follows that we cannot have $p_i \in (p_j - t, p_j + t)$ in a pure strategy Bertrand Nash equilibrium. ■

2.1 Rational Rationing in a Spatial Market

Given the payoff structure of a spatial market, rational rationing uniquely defines each buyer's payoff for a given p and G. To see this note that the sellers in G always have excess capacity and as such for all $q \in Q^r(p, G)$ and $b_i \in B$,

$$q_i = \underset{q_i}{\operatorname{argmax}} \int_0^{\sum_{j\in S} q_{ij}} D_i^{-1}(\vartheta)d\vartheta - \sum_{j\in S}\big(\mathbb{1}_{q_{ij}>0}K_{ij} + q_{ij}(p_j + k_{ij})\big) \qquad (2.4)$$

Any q that violates this condition for a buyer b_i must also violate definition 1.

Imposing the requirement for rational rationing delivers us the following general result:

Lemma 2. *Consider a market in which buyer and seller payoffs satisfy* (2.1) *and* (2.2)*. For all $p \in \mathbb{R}^{|S|}$, $G \subseteq N$, $q \in Q^r(p, G)$ and $b_i \in G$, if $q_{im} > 0$ then,*

$$\int_0^{\sum_{j\in S} q_{ij}} D_i^{-1}(\vartheta)d\vartheta - \sum_{j\in S}\big(\mathbb{1}_{q_{ij}>0}K_{ij} + q_{ij}(p_j + k_{ij})\big)$$
$$= \int_0^{\hat{q}_{im}} D_i^{-1}(\vartheta)d\vartheta - \big(K_{im} + \hat{q}_{im}(p_m + k_{im})\big), \quad (2.5)$$

where $\hat{q}_{im} = \sum_{j\in S} q_{ij}$. Moreover, there exists a $\hat{q} \in Q^r(p^*, q^*)$ such that $\hat{q}_i = \{0, \ldots, 0, \hat{q}_{im}, 0, \ldots, 0\}$.

Lemma 2 states that given a price vector p, each buyer in a coalition can pick any seller in the coalition with whom they would be willing to trade some strictly positive amount, and still maximise his payoff by trading only with that seller. This result is straightforward once you recognise that the products being traded in the market are homogeneous save for their spatial characteristics, and that the transport costs of dealing with any given seller are a (weakly) concave function of the quantity traded. The proof of lemma 2 can be found in appendix A.

The significance of this result is that we need only consider deviating coalitions with a single seller when constructing the core, which gets us half way to proving that the core is defined by a single condition on each seller's price. The full significance of lemma 2 is made clear in the proof of proposition 1 below.

Example 2.2. Returning to the switching costs example, rational rationing admits a very limited range of trades. Given the payoff structure $q \in Q^r(p, N)$ implies,

$$(q_{ii}, q_{ji}) \in \begin{cases} \{(0,0)\} & p_i > p_j + t \\ \{(0,0),(1,0)\} & p_i = p_j + t \\ \{(1,0)\} & p_j - t < p_i < p_j + t\ , \\ \{(1,0),(1,1)\} & p_i = p_j - t \\ \{(1,1)\} & p_i < p_j - t \end{cases} \quad \forall i \neq j. \tag{2.6}$$

In other words, when all players are party to a transaction trades are uniquely defined by price except where two sellers "tie" in the offer that they present to a buyer. In this case that buyer becomes indifferent between the two sellers but will not split his purchase as this would cause the buyer to incur the full transport cost t regardless of the quantity that he receives from the distant seller. ■

2.2 The Undercut-Proof Criteria

The core of a constrained coalitional game, in a spatial market, is remarkably well behaved. Hamilton et. al. (1991) have shown that on the Hotelling line with a continuum of buyers an allocation lies in the core of (N, V) if and only if it can be generated by *undercut-proof* prices. The following proposition extends this result to our generalised spatial setting.

Proposition 1 (Undercut-Proof Criteria)**.** *Consider a market in which buyer and sellers payoffs satisfy* (2.1) *and* (2.2)*. An allocation x^* lies in the core of (N, V) if and only if $x^* = x(p^*, q^*)$ where $q^* \in Q^r(p^*, N)$ and the pair (p^*, q^*) satisfies,* *i.* $p_j^* \geq c_j$ *for all* $s_j \in S$*;*

ii. And for all s_j *and* $p_j < p_j^*$*, if* $p_j^* > c_j$ *and there exists* $\bar{p}_j > c_j$ *and* $\bar{q} \in Q^r\big((p_{-j}^*, \bar{p}_j), N\big)$ *such that* $\bar{q}_j \neq \mathbf{0}$*, then,*

$$x_j(p^*, q^*) > (p_j - c_j) \sum_{i \in B} q_{ij}, \tag{2.7}$$

for all $q \in Q^r\big((p_{-j}^*, p_j), N\big)$.

The proof of proposition 1 can be found in appendix B, however a brief outline is presented here. The proof consists of two elements: First, it is necessary to show that if a vector of payoffs cannot be implemented by an undercut-proof price vector then it does not lie in the core. This is straightforward as the coalition consisting of the deviating seller, and all buyers willing to purchase her product at the lower price, can replicate the gains that they achieve through a unilateral price drop in a deviating coalition that agrees to the same lower price.

The second part of the proof is to demonstrate that if a coalition of players can block a vector of payoffs then that vector of payoffs cannot be implemented by an undercut-proof price vector. If a blocking coalition contains a single seller and at least one buyer, the logic is the reverse of that outline in the previous paragraph. Where a deviating coalition contains more than one seller we know from lemma 2 that we can reduce this coalition to a sub-coalition consisting of a single seller and all the buyers who traded with this seller in the original deviating coalition. This sub-coalition must also be able to block the vector of payoffs as every member will be at least as well off as they were in the original deviation.

The undercut-proof criteria readily extend to the case in which a seller occupies multiple locations and is able to set a different price at each location. Where a seller $s_j \in S$ occupies m locations, s_j's price vector can be expressed as $p_j = (p_{j1}, \dots, p_{jm})$. In this case if we read $p_j < p_j^*$ as $p_{jn} \leq p_{jn}^*$ for all $n \in 1, \dots, m$, with strict inequality for some n, proposition 1 continues to hold.

Lemma 2 plays a key role in the proof of proposition 1 as it allows us to focus only on deviating coalitions containing a single seller, and from there to reduce the definition of the core to a single condition on each sellers price. However, lemma 2 only applies where each buyer's payoff is independent of the quantities consumed by the other buyers. The undercut-proof criteria does not generalise to the case in which there are externalities from consumption.

The other critical assumption in the spatial model is the unlimited capacity of the sellers. Unlimited capacity eliminates competition between buyers, giving them a common preference for lower prices. In the capacity constrained model examined in the next section upward pressure on prices may arise where capacity is insufficient to satisfy demand at a given price.

Proposition 1 provides a rigourous technical justification for Shy's (2001) approach. Moreover, the undercut-proof criteria dramatically simplifies the task of defining the core as it allows the problem to be solved without reference to outcomes that can be generated by the $2^N - 1$ potential deviating coalitions. This ease of use is apparent in the solution to the switching costs model.

Example 2.3. The core of the switching costs model is defined with respect to 15 distinct feasible sets corresponding to the 15 deviating coalitions that are possible in a four player game. By employing proposition 1 this process can be reduced to one of identifying the market share that can be supported in the core, before deriving the undercut-proof condition on the two prices. By proposition 1 prices that satisfy this condition must implement allocations in the core.

First note that proposition 1 allows us to assume that $p_i \geq c$ for all $i \in \{1, 2\}$. It is straightforward to see that each buyer must trade with their closest seller. For it to be the case that s_i does not trade with b_i, we must have $p_i \geq p_j + t$ and $x_{s_i} = 0$.[15] In this situation s_i can strictly increase her payoff by reducing her price to $p_i' = p_j + t - \varepsilon$ for sufficiently small $\varepsilon > 0$.

To complete the characterisation of the core we need only determine the conditions under which prices are undercut-proof given this division of the market. Trading only with her collocated buyer the seller s_i receives the

[15] This follows directly from the result on rational rationing developed in example 2.2.

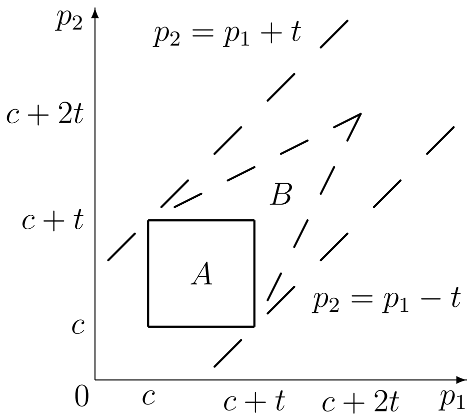

Figure 3: Core Prices

payoff,

$$x_{s_i} = p_i - c. \tag{2.8}$$

The greatest payoff that s_i can achieve through a deviation occurs when s_i undercuts s_j's price by an amount t and $q_i = (1, 1)$. In this case s_i receives the payoff,

$$x_{s_i} = 2(p_j - t - c). \tag{2.9}$$

Undercut-proofness requires the payoff in (2.8) to be strictly greater than the payoff in (2.9). Rearranging we get,

$$p_j - c \leq \frac{p_i - c}{2} + t, \tag{2.10}$$

with strict inequality where $p_i > c$. Solving (2.10) simultaneously for all $i \in \{1, 2\}$ yields $\sup p = (2t + c, 2t + c)$. The range of prices that implement core allocations is illustrated in figure 3 as union of the areas A and B. Note that both the upper and rightward boundaries are not included.

The area AB lies between the lines $p_2 = p_1 - t$ and $p_2 = p_1 + t$. These lines indicate the prices at which buyers become indifferent between the two sellers. The only core allocations that leave a buyer indifferent between the two sellers correspond to the prices $p = (c + t, c)$ and $p = (c, c + t)$. These allocations still require each buyer to trade exclusively with their local seller, however the prices remain undercut-proof as the low price seller is trading at cost and has no incentive to expand her market share.

The switching costs model provides a good illustration of the way in which constraining the set of feasible payoffs can expand the core. The core of a TU analysis of this game yields each seller a payoff in the interval $[0, t]$ as the marginal contribution of each seller to a TU analysis of this model is the travel cost t that she eliminates for her collocated buyer. The set of prices that are implied by these payoffs are illustrated in figure 3 as the region A.

The prices in B correspond to allocations that do not lie in the core of a TU analysis of the switching cost model, but do lie in the core of the constrained coalitional price setting game. Sellers benefit from their inability to selectively target discounts to individual buyers. A seller cannot attract the distant buyer without dropping her price, this in turn reduces the revenue that she receives from her local buyer. Sellers are less willing to discount in the constrained coalitional price setting game and this reluctance is illustrated by the inclusion of the region B in the set of prices that implement core allocations. ■

3 Price Competition with Convex Costs

Edgeworth (1925) established that the Bertrand price setting game may fail to possess a pure strategy Bertrand Nash equilibrium where goods are homogeneous and sellers face convex cost functions.[16] This fact can be clearly seen in Kreps and Scheinkman's (1983) proof that Cournot competition is equivalent to Bertrand competition with capacity pre-commitment. In Kreps and Scheinkman's model a pure strategy Bertrand Nash equilibrium fails to exist where sellers commit to capacities that are greater than their respective Cournot best responses.

[16] To be precise Edgeworth (1925) — writing before the concept of a mixed strategy had been developed — argues that the equilibrium may be *indeterminate* where sellers face convex costs. Instead, Edgeworth conjectured that faced with convex costs sellers would progressively undercut one and other until the profits that can be gained by undercutting rivals are less than those available to a seller by setting the monopoly price against residual demand. This would give rise to a cycle in which each period of undercutting would be followed by a sharp price rise.

Price competition with convex costs poses a greater challenge for the constrained coalitional price setting game than was the case with the spatial model developed in the previous section. Convex costs may result in sellers facing an upper bound on the quantity that they are willing to supply at a given price and may result in demand exceeding supply at some prices. This forces us to consider a new factor in our analysis, that of competition between buyers for scarce supply.

Three results are developed in this section. To begin with we show that all sellers setting the market clearing price always implements a core outcome. In an associated example it is shown that both monopoly and monopsony power have the ability to admit inefficient outcomes into the core. Moreover, monopoly and monopsony power are treated symmetrically in this model. Finally, we demonstrate that where demand is coarse the difficulty of neatly matching buyers and sellers in a deviating coalition may admit inefficient outcomes even where there exists competition on both sides of the market. However, we are able to show that if demand is sufficiently fine the core collapses to the market clearing outcome. The proof of this final result is both inspired by, and similar to, the core equivalence proof of Debreu and Scarf (1963).

We continue to employ a very general structure in this section. Each seller's cost function is assumed to conform to the conditions $C_j' \geq 0$ and $C_j'' \geq 0$, while buyer payoffs are assumed to take the form,

$$x_i(p,q) = \int_0^{\sum_{j \in S} q_{ij}} D_i^{-1}(\vartheta) d\vartheta - \sum_{j \in S} p_j q_{ij}. \tag{3.1}$$

For the purposes of analysing this model it is useful to aggregate demand and supply functions. For a coalition $G \subseteq N$ let,

$$D_G(\rho) = \sum_{i \in B \cap G} D_i(\rho), \tag{3.2}$$

and let $D_G^{-1}(\cdot)$ be the inverse of this aggregate demand. Moreover defining,

$$Y_j^{-1}(\vartheta) = \frac{d}{d\vartheta} C_j(\vartheta) \tag{3.3}$$

to be the inverse of s_j's supply function we can write aggregate supply as,

$$Y_G(\rho) = \sum_{j \in S \cap G} Y_j(\rho), \tag{3.4}$$

and the inverse of this aggregate supply as $Y_G^{-1}(\cdot)$.

3.1 Market Clearing Prices and the Core

A market clearing price is a price at which neither excess demand nor excess supply exists in a market. In this section we prove that each seller setting a price equal to the market clearing price always implements an allocation in the core of the constrained coalitional price setting game. However, inefficient allocations may also lie within the core where competition is absent on at least one side of the market.

Given the nature of our analysis it is necessary to define a market clearing price for every $G \subseteq N$. Formally, define p_G^c implicitly as the set of prices that satisfies,

$$p_G^c = \big\{\rho \in \mathbb{R} : D_G(\rho) = Y(\rho)\big\}. \tag{3.5}$$

Given the assumptions placed on the form of D and C, p_G^c must be a convex set.

Proposition 2. *Suppose that buyer payoffs satisfy* (3.1)*, and that seller cost functions satisfy* $C_j' \geq 0$ *and* $C_j'' \geq 0$*. If* p^* *satisfies* $p_i^* = p_j^* \in p_N^c$ *for all* $i, j \in S$*, and* $q^* \in Q^r(p^*, N)$*, the allocation* $x(p^*, q^*)$ *lies in the core of* (N, V)*.*

The proof to proposition 2 can be found in appendix C. The structure of the proof is straightforward: It is shown that an allocation that can be implemented by each seller setting the market clearing price must lie in the core of the TU analysis of the convex costs model. Given that this allocation is feasible to the grand coalition, by lemma 1 this allocation must also lie in the core of the constrained coalitional price setting game.

A direct consequence of proposition 2 is that there must always exist at least one efficient allocation in the core of the constrained coalitional price setting when it is applied to a market in which the sellers face convex costs.

This guarantees both that the core will be non-empty, and that there must exist at least one efficient outcome that is also stable.

However, proposition 2 does not exclude the possibility that we may also find inefficient outcomes in the core. The following example demonstrates that distortions may be created by both monopoly and monopsony power.

Example 3.1. Consider a market in which a monopoly seller is trading with a monopsony buyer. Let $N = \{b, s\}$ with,

$$x_b(p, q) = q(1 - p) - \frac{q^2}{2}, \tag{3.6}$$

and,

$$x_s(p, q) = qp - \frac{q^2}{2}. \tag{3.7}$$

These payoffs correspond to the demand function $D(p) = 1 - p$, and the supply function $Y(p) = p$. Given (3.6) and (3.7) rational rationing implies that Q^r is a singleton taking the value,

$$Q^r(p, N) = \min\big\{p, (1 - p)\big\}, \tag{3.8}$$

which is to say the quantity traded is limited by the lesser of monopoly supply and monopsony demand at the prevailing price. The market clearing price is $p_N^c = 1/2$.

We claim that an allocation $x(p^*, q^*)$ lies in the core of (N, V) if and only if $p^* \in [1/3, 2/3]$. To see this note that individually b and s can only guarantee themselves a payoff zero. Given that rational rationing also guarantees each player a non-negative payoff, it follows that an allocation x can only be blocked by the coalition N. For $q \in Q^r(p, N)$, each player's payoff is concave in p with (3.6) achieving a maximum at $p = 1/3$ while (3.7) is maximised at $p = 2/3$. For $p < 1/3$ and $p > 2/3$ players have a common interest in moving the price toward the interval $[1/3, 2/3]$, however for $p \in [1/3, 2/3]$ there is a tension between the preferences of the buyer and seller and they will be unable to construct a mutually beneficial deviation.

This example is significant for two reasons. First, it demonstrates that when transactions are constrained to take place at linear prices, it is possible for inefficient outcomes to accompany the efficient outcome in the core. A player who controls one side of the market has the capacity to exploit this position by pushing the price away from the market clearing level. This captures the player a greater payoff, while at the same time diminishing the total value created in the market. The distortion is identical to that created by a monopoly in the Bertrand price setting game. That the inefficiency can be sustained in the core of the constrained coalitional price setting game illustrates the importance of those allocations that are possible in a TU analysis, but are absent from the feasible set $V(N)$.

Second, this example demonstrates that market power is treated symmetrically regardless of on which side of the market it arises. The monopoly price is one possible outcome here, while at the same time if the monopsony buyer possesses a greater degree of "bargaining power" than the seller, we would expect the price to lie below a market clearing level. ■

3.2 Collapsing the Core

The final result for this section concerns the conditions under which the core collapses to only admit allocations corresponding to market clearing prices. We begin with an example that demonstrates the way in which a coarse division of demand amongst buyers can give rise to inefficient core outcomes before proving that when demand is sufficiently fine the core only admits market clearing outcomes.

The core of the constrained coalitional game is defined with respect to outcomes that can be generated when some subset of players are excluded from the game. In the spatial model excluding a group of players was not problematic as buyers were happy to trade exclusively with their preferred seller, and sellers had unlimited capacity. However, in the presence of convex costs a single seller may not be able to satisfy the demand of a group of buyers, and as such the buyers my be unwilling to exclude every other seller from a deviating coalition.

Inefficient core outcomes emerge as possibility where there exists a difficulty in matching buyer demand to seller supply in a deviating coalition. This source of inefficiency is distinct from the monopoly distortion examined in example 3.1, and cannot be attributed to a lack of competition. The problem can be clearly seen in the following example:

Example 3.2. Consider a market with two buyers and two sellers. Suppose that buyer 1 faces the demand function $D_1(p) = 1-p$, while buyer 2's demand is given by $D_2(p) = 2(1-p)$. Similarly, suppose that seller 1's supply function is $Y_1(p) = p$, while for the second seller $Y_2(p) = 2p$. In this market $p_N^c = 1/2$.

Rational rationing allows for the price vector $p = (2/3, 1/2)$ to result in s_1 trading 1/3 of a unit to b_1, while b_2 purchases one unit from s_2. This outcome is clearly not efficient as one of the trades does not take place at the market clearing price, however we claim that it is a core outcome. Proving this can be accomplished by demonstrating that no player can strictly benefit from a deviation.

Seller 1: Seller 1 has set the monopoly price relative to buyer 1's demand. For seller 1 to improve her payoff she must sell some quantity to buyer 2 at a price that is greater than 1/2. However, buyer 2 is already satisfying his demand at the price $p_2 = 1/2$ and therefore would never be a party to this deviation.

Buyer 2: Buyer 2 cannot benefit unless one or both of the sellers lowers her price below 1/2, however such a deviation must leave that seller strictly worse off.

Seller 2: Selling one unit at $p_2 = 1/2$ yields seller 2 a payoff of 1/4, exhausting her capacity in the process. It follow that seller 2 cannot increase her payoff without raising her price. Buyer 2 would not be a party to a price increase and as such seller 2 can only trade with buyer 1 in a deviation that increases her payoff. Against buyer 1's demand, seller 2's monopoly price is $p_2 = 3/5$. While this price would be acceptable to buyer 2 as it is less that the price $p_1 = 2/3$ that he is currently paying, it would only yield seller 2 a payoff of 1/5 which is less than her payoff under the original price vector.

Buyer 1: Buyer 1 cannot benefit without seller 1 lowering her price — which she would not be willing to do — or being able to purchase some part of seller 2's output at $p_2 = 1/2$. This second option would reduce the allotment going to buyer 2, which would prevent his participation, and seller 2 will not deviate without buyer 2. ■

The distortion illustrated in this example is clearly not the result of a lack of competition. Rather, it occurs because buyer 2 and seller 2 are larger than their rivals, and cannot easily be matched with them in a deviating coalition. Many coalitional games experience similar difficulties when faced with asymmetric players and these asymmetries can substantially increase the size of the core. One method for eliminating surplus allocations from the core was developed by Debreu and Scarf (1963) in the development of the core equivalence theorem.

The *core equivalence theorem* compares the core of a TU coalitional analysis of an economy with its Walrasian allocation. It is readily shown that Walrasian allocation lies in the core of this TU game. However, in the presence of asymmetries the core can also contain many non-Walrasian allocations. Following Edgeworth's (1925) conjecture, Debreu and Scarf replicate the economy n times, showing that the only allocation that lies in the core of the n-replica of the TU game, for all $n \in \mathbb{N}$, is the Walrasian allocation.

The approach adopted in this paper differs from that adopted by Debreu and Scarf (1963) in one important respect: Instead of replicating the entire economy we divide each buyer into n equal parts, while leaving sellers untouched. The intuition that we wish to develop is that when there exists competition between sellers, as demand becomes fine relative to supply, the core of the constrained coalitional price setting game will converge to the market clearing outcome.

Definition 3. The n-slice of the game (N, V) is defined as the game (N_n, V) in which each buyer has been *sliced* into n equal parts. Formally, the grand coalition becomes $N_n = S \cup B^n$ where $B^n = \{b_{i1}, \ldots, b_{in}\}_{i \in B}$. The payoff to each buyer continues to satisfy (3.1) with $D_{ik}(\rho) = \frac{1}{n} D_i(\rho)$ for all $i \in B$ and $k \in \{1, \ldots, n\}$.[17] To complete the notation we define q_{ikj} as the quantity that buyer b_{ik} purchases from seller s_j. Note that $p^c_N = p^c_{N_n}$ for all $n \in \mathbb{N}$.

In the interests of tractability will assume that buyer demand is twice continuously differentiable and strictly downward sloping. Moreover we assume that every seller is willing to supply a strictly positive quantity at the

[17] Equivalently $D^{-1}_{ik}(\frac{\vartheta}{n}) = D^{-1}_i(\vartheta)$.

market clearing price and that removing any single seller would alter the market clearing price. These assumptions are not necessary but aid considerably in restricting the length of an already complex proof. Amongst other things, these additional assumptions guarantee that the market clearing price will be unique, and prevent the inclusion of sellers who have no impact upon the game.

Proposition 3 (Core Convergence)**.** *Suppose that buyer payoffs satisfy* (3.1) *with D_i twice continuously differentiable and strictly downward sloping for all $i \in B$. Suppose further that $|S| \geq 2$ and that $C_j' \geq 0$, $C_j'' \geq 0$, $Y_j(p_N^c) > 0$ and $p_N^c \neq p_{N-j}^c$ for all $j \in S$.*

Consider the price vector p^*. If there exists $j \in S$ such that $p_j^* \neq p_N^c$, then there exists at least one $n \in \mathbb{N}$ such that,

$$x(p^*, q^*) \notin \text{Core}(N_n, V), \tag{3.9}$$

for all $q^* \in Q^r(p^*, N_n)$.[18]

The proof of proposition 3 is set out in appendix D but a brief outline is presented here. The proof is divided into three steps: First it is shown that if a vector of prices can implement an allocation in the core of (N_n, V) for all $n \in \mathbb{N}$, all sellers must set the same price. Second we show that prices that leave excess demand cannot implement core allocations for all n. And finally we show that prices that leave excess supply likewise cannot implement core allocations for all n. As each price vector that violates proposition 3 must violate one of these conditions we have our proof.

Of the three steps the first is the most technically challenging. In order to prove that sellers cannot trade at different prices we take a coalition consisting of the lowest priced seller, a buyer who purchases from the highest priced seller, and other randomly assigned buyers. We require that the total demand of the buyers be at least as large as the seller's willingness to supply at the low price, and that by subtracting the difference between the first

[18]In fact for any non-compliant price vector we can find an infinite and strictly increasing sequence of natural numbers for which the price vector cannot implement an allocation in the core of (N_n, V).

buyer's demand at the low and high prices we are left with a total demand that is strictly less than the seller's willingness to supply. By agreeing to the low price this deviating coalition can strictly increase the payoff to the first buyer without leaving any other player worse off.

Proving that there always exists an n-slice for which such a coalition can be constructed turns out to be equivalent to proving that there always exists natural numbers m and n such that,

$$\frac{1}{n}\varepsilon > \frac{m}{n} - \delta \geq 0, \tag{3.10}$$

for arbitrary $\varepsilon > 0$ and $\delta \in (0, 1)$. The problem is complicated by the fact that n appears in the first two terms of the equation. Consequently, as n is increased to allow a tighter fit, the size of the interval into which the term $\frac{m}{n} - \delta$ must be confined, also contracts.

Once we are able to assume that all trades must occur at the same price the remaining two steps are straightforward. Prices cannot leave buyers with excess demand as for a sufficiently fine slice a successful deviation could be constructed by excluding a small group of buyers, redistributing the quantities that they would have received amongst buyers with excess demand. Excess supply can similarly be excluded as a possibility given that when demand is sufficiently fine a seller with excess capacity can gain by undercutting her rivals.

The following example illustrates this three step process. In this example we employ a continuum of buyers, thereby avoiding the matching problems that were illustrated in example 3.2.[19]

Example 3.3. Consider a market with a unit mass of buyers, each of whom demands one unit of an indivisible good and has a willingness to pay of one. Suppose that there are two sellers present in this market and that both sellers have a marginal cost of zero and a common capacity of $\bar{q}$. We will consider cases in which $\bar{q}$ belongs to the open interval $(0, 1)$.

From proposition 2 we know that market clearing prices implement core allocations. In other words their exists $q^* \in Q^r(p^*, N)$ such that the allocation $x(p^*, q^*)$ lies in the core if $p_1^* = p_2^*$ and $p_1^* \in p_N^c$. Given that the total

[19]You can think of this model as having one buyer sliced into uncountably many parts.

capacity of sellers in the market is $2\bar{q}$ we have,

$$p_N^c = \begin{cases} \{1\} & \bar{q} \in (0, 1/2) \\ [0, 1] & \bar{q} = 1/2 \\ \{0\} & \bar{q} \in (1/2, 1) \end{cases}. \tag{3.11}$$

It only remains to show that no other allocation lies in the core. Three steps allow us to eliminate all other allocations as possibilities.

Step 1: For $x(p^*, q^*)$ to lie in the core we must have $p_1^* = p_2^*$. Suppose to the contrary that $p_1^* < p_2^*$, and consider the coalition of seller 1 and a mass $\bar{q}$ of buyers containing every buyer who purchases their unit from seller 2 in q^*. By agreeing to the price p_1^* this coalition strictly increases the payoff of the buyers who purchased from seller 2 without reducing the payoff to any other member of the coalition.

Step 2: A core allocation cannot leave a buyer with excess demand, implying that the common price charged by sellers cannot drop below 1 where $\bar{q} \in (0, 1/2)$. Suppose to the contrary that $p_1^* = p_2^* < 1$. There must exist a mass $1 - 2\bar{q} > 0$ of buyers who do not trade. Consider a coalition consisting of both sellers and a mass $2\bar{q}$ of buyers containing a strictly positive mass of buyers who do not trade. By retaining the original price this coalition can strictly increase the payoff to the buyers who were not originally allocated a unit, without reducing the payoff to any other member of the coalition. This deviation cannot block an allocation when $p_1^* = p_2^* = 1$ as buyers do not benefit from increasing the amount that they purchase when the price is equal to their valuation.

Step 3: A core allocation cannot leave a seller with excess capacity, consequently sellers cannot lift their price above zero where $\bar{q} \in (1/2, 1)$. Suppose to the contrary that $p_1^* = p_2^* > 0$ and note that at least one seller must be trading a quantity that is less than their capacity $\bar{q}$. Consider the coalition of one such seller and a mass $\bar{q}$ of buyers. For sufficiently small $\varepsilon > 0$ every member of this coalition can strictly increase their payoff by agreeing to the price $p_1^* - \varepsilon$. ■

This model inspired Edgeworth's (1925) original criticism of the Bertrand price setting game. Edgeworth showed that where $\bar{q} \in (1/2, 1)$ the model does not possess a pure strategy Bertrand Nash equilibrium. Moreover, the support of the mixed strategy Bertrand Nash equilibrium is bounded away from the market for these capacities.

Contrast this with the constrained coalitional price setting game. Proposition 2 guarantees that we always get a market clearing outcome in the core, while proposition 3 and example 3.3 demonstrate that there is a sense in which these market clearing outcomes can be regarded as more robust than any other allocation that may lie in the core of a particular example. The only indeterminacy in the constrained coalitional price setting game arises where p_N^c is itself an interval.

The presence of market clearing outcomes in the core of the constrained coalitional price setting game gains a particular significance when considering Cournot's model of quantity competition. In Cournot competition it is assumed that sellers unilaterally select output levels before this output is auctioned off at the market clearing price. Kreps and Scheinkman (1983) proved that the Cournot Nash equilibrium is equivalent to the Bertrand Nash equilibrium if sellers must unilaterally commit to a capacity in advance of the market. The result does not hold for out of equilibrium outcomes as where the capacities selected by sellers are greater than Cournot best responses a pure strategy Bertrand Nash equilibrium does not exist and expected payoffs from the mixed strategy equilibria are greater than market clearing payoffs.

Propositions 2 and 3 show that the equivalence between Cournot competition, and price competition with capacity pre-commitment, is completed if prices are determined in accordance with the constrained coalitional price setting game.

4 Vertical Contracting with a Linear Price Constraint

The final model considered in this paper is price formation in vertically related markets. Specifically, we examine the case in which the output produced by sellers must be transformed by an *intermediary* in order to be considered valuable by buyers. For the purposes of this section the products in both the upstream and downstream markets are assumed to be homogeneous, and transformation is assumed to occur at a constant $1:1$ ratio.

Two results are developed here. The first, an extension of the market clearing result developed in section 3, shows that it is always a core outcome for both markets to operate at market clearing prices. The second demonstrates that double-marginalisation, as generally understood in the literature, is never supported as a core outcome. However, where the costs of both seller and intermediary are strictly convex, a vertical merger still has the potential to produce a small social benefit.

Intermediaries are at the same time buyers in the upstream market and sellers in the downstream market. Like sellers, each intermediary is constrained to set a single unit price for his output, and is unable to differentiate between buyers in the downstream market. Let M be the set of intermediaries who are capable of transforming the upstream product, for sale in the downstream market. For each $m_k \in M$ let $C_k(\vartheta)$ denote the cost of transforming the quantity ϑ. We assume that $C_k' \geq 0$ and $C_k'' \geq 0$ for all $k \in M$.

In order to completely specify transactions we must extend p and q to capture the transactions in both the upstream and downstream markets. Formally, let $p \in \mathbb{R}^{|M \cup S|}$ denote the unit price charged by each seller and intermediary for their output, and let $q \in \mathbb{R}_+^{|B \times M \cup M \times S|}$ denote the quantity exchanged between any buyer-intermediary and intermediary-seller pair. In order to distinguish sellers and intermediaries we define,

$$T_k^{-1}(\vartheta) = \frac{d}{d\vartheta} C_k(\vartheta) \tag{4.1}$$

to be the inverse of m_k's supply function. The aggregate supply of transformations is expressed as,

$$T_G(\rho) = \sum_{k \in M \cap G} T_j(\rho), \tag{4.2}$$

and the inverse of this aggregate supply as $T_G^{-1}(\cdot)$.

Intermediary payoffs take the same basic form as seller payoffs. For each intermediary $m_k \in M$ this payoff can be expressed as,

$$x_k(p,q) = \sum_{i \in B} p_k q_{ik} - C_k\Big(\sum_{i \in B} q_{ik}\Big) - \sum_{j \in S} p_j q_{kj}, \tag{4.3}$$

subject to,

$$\sum_{i \in B} q_{ik} \leq \sum_{j \in S} q_{kj}. \tag{4.4}$$

This last inequality states that the total quantity purchased by an intermediary in the upstream market must be at least as large as the quantity that he produces for the downstream market. Rational rationing implies that (4.4) holds with equality.

4.1 Efficiency and the Core

In moving to a vertical setting with a fixed transformation ratio, it is convenient to switch our focus from prices to quantities traded. For example it is easiest to define the market clearing outcome in terms of aggregate quantity traded, only then backing out the market clearing prices.

The value created by a coalition G is maximised when the total quantity traded in each market q_G^c satisfies,

$$D_G^{-1}(q_G^c) = Y_G^{-1}(q_G^c) + T_G^{-1}(q_G^c). \tag{4.5}$$

We write $p_G^d = D_G^{-1}(q_G^c)$ to denote the market clearing price in the downstream market, and $p_G^u = p_G^d - T_G^{-1}(q_G^c)$ to denote the upstream market clearing price.

Proposition 4. *Suppose that buyer payoffs satisfy* (3.1)*, and that intermediary and seller cost functions satisfy* $C_j' \geq 0$ *and* $C_j'' \geq 0$*. For* p^* *satisfying* $p_i^* = p_j^* \in p_N^u$ *for all* $i, j \in S$ *and* $p_k^* = p_l^* \in p_N^d$ *for all* $k, l \in M$*, and* $q^* \in Q^r(p^*, N)$*, the allocation* $x(p^*, q^*)$ *lies in the core of* (N, V)*.*

The proof of proposition 4 can be found in appendix E. The proof is a relatively straightforward extension of the proof of proposition 2 and we will not dwell on it here. Save to say that the logic of the proof can be extended iteratively to n vertically related markets, demonstrating that efficiency is always a stable outcome.

4.2 Double-Marginalisation

One of the most commonly quoted social welfare rationales for vertical mergers, in markets where both the seller and intermediary are monopolists, is that a vertical merger eliminates double-marginalisation. Double-marginalisation occurs in the Bertrand price setting game because both the seller and the intermediary act unilaterally, each imposing a distortion on the market in order to extract rents. The collective effect of these distortions is greater that the distortion created by a vertically integrated monopolist. From an anti-trust point of view, the irony of double-marginalisation is that both profit and social welfare would be increased if the seller and intermediary could co-ordinate their prices.

In this subsection we show that whilst the presence of a monopolist at any of the three levels of the market may admit inefficient outcomes to the core of the constrained price setting game, these inefficiencies cannot exceed that which would be created were any one of the monopoly players able to dictate the price in both markets. This excludes the possibility of double-marginalisation arising as a stable outcome of the constrained coalitional price setting game. However, a weaker form of double-marginalisation may sill occur where both the seller and intermediary face strictly convex cost functions.

We begin with a simple example that allows us to compare the maximum distortion that can arise in a vertically related market under both Bertrand and constrained coalitional price setting. Specifically, for a simple model we develop the double-marginalisation result that arises in Bertrand price setting and show that this outcome can be blocked in the constrained coalitional price setting game.

For the purposes of this subsection we assume that each level of the market is occupied by a monopolist and as such $N = \{b, m, s\}$. The prices in the up and downstream markets are denoted p^u and p^d respectively, and q is the quantity traded in both markets.

Example 4.1. Consider a pair of vertically related markets in which both the seller and the intermediary face a constant marginal cost of zero, while the

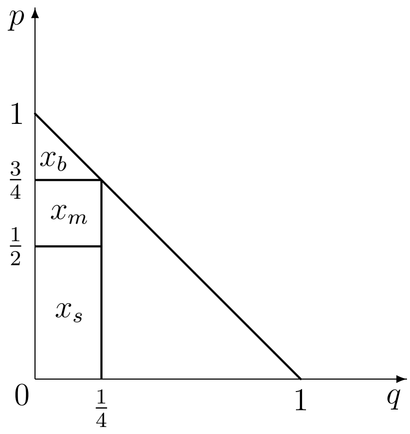

Figure 4: Double-Marginalisation

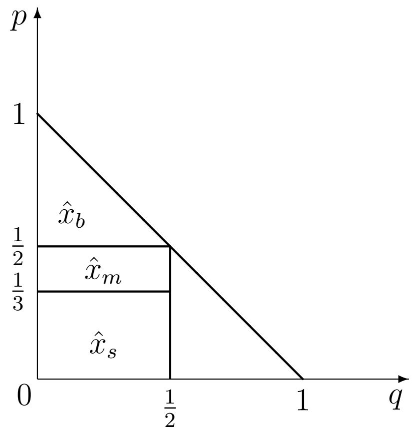

Figure 5: Co-ordinated Distortion

buyer's demand is given by $D(p^d) = 1 - p^d$.

In the standard Bertrand treatment price setting is sequential with the seller setting a price for her output before the intermediary sets the price for the downstream market. In a sub-game perfect Bertrand Nash equilibrium the intermediary selects p^d to maximise his profit given the upstream price;

$$p^d = \underset{\rho}{\text{argmax}}(\rho - p^u)(1-\rho) = \frac{1+p^u}{2}. \tag{4.6}$$

Similarly, the seller selects p^u such that,

$$p^u = \underset{\rho}{\text{argmax}}\, \rho\Big(1 - \frac{1+\rho}{2}\Big) = \frac{1}{2}, \tag{4.7}$$

which in turn implies $p^d = 3/4$ and that the total quantity traded $q = 1/4$. This outcome is illustrated in figure 4 and produces the payoffs $x_s = 1/8$, $x_m = 1/16$ and $x_b = 1/32$.

The double-marginalisation outcome is both highly inefficient and fails to maximise total industry profits; the sum of seller and intermediary payoffs. All three players could be made better off by a co-ordinated price drop resulting in $\hat{p}^d = 1/2$ and $\hat{p}^u = 1/3$. At $\hat{p}^d = 1/2$ the quantity traded is twice the double-marginalisation level and industry profits are maximised. While still inefficient, the prices $\hat{p}^d = 1/2$ and $\hat{p}^u = 1/3$ deliver a strict Pareto improvement over the double-marginalisation outcome, resulting in the payoffs $\hat{x}_s = 1/6$, $\hat{x}_m = 1/12$ and $\hat{x}_b = 1/8$. This outcome is illustrated in figure 5.

The existence of a Pareto improvement that can be achieved via a co-ordinated price movement does not prevent the double-marginalisation outcome from being an equilibrium of the Bertrand price setting game as a Nash equilibrium is defined with respect to unilateral deviations only. However, the constrained coalitional price setting game does allow players to co-ordinate price movements and consequently the strong double-marginalisation outcome cannot be a core outcome in this example. ■

Where the seller and intermediary both face a constant marginal cost the upstream market price can be used to arbitrarily assign industry profits between these two players. In the context of the constrained coalitional price setting game this means that it is only the sum of seller and intermediary payoffs that matter as these players are not constrained in how they divide their joint surplus. Given that the buyer always benefits from a drop in the upstream price, it follows that an allocation cannot lie in the core if the quantity traded is less than the quantity that maximises total industry profits.

The situation is more complicated if either the seller or intermediary face convex costs. The existence of convex costs limits the extent to which value can be transferred between seller and intermediary via the upstream market price. Seller and intermediary preferences diverge to the extent that the best feasible outcome for each player may arise at different quantities. Nevertheless, for relatively well behaved payoff functions we can demonstrate that the quantity traded in a core allocation cannot be less than the quantity that either the the seller or intermediary would set were they able to dictate the price in both markets. This result is stated formally in proposition 5 below.

Before proceeding to the proof it is worth establishing which outcomes would be preferred by the seller and intermediary. If the seller could dictate prices in both markets she would choose to trade a quantity satisfying,

$$q_s^* = \underset{q}{\operatorname{argmax}}\Big(p^u q - \int_0^q Y^{-1}(\vartheta)d\vartheta\Big), \tag{4.8}$$

where $p^u = D^{-1}(q) - T^{-1}(q)$. Similarly, were the intermediary able to dictate the price in both markets me would choose to trade the quantity,

$$q_m^* = \underset{q}{\operatorname{argmax}}\Big((p^d - p^u)q - \int_0^q T^{-1}(\vartheta)d\vartheta\Big), \tag{4.9}$$

where $p^d = D^{-1}(q)$ and $p^u = Y^{-1}(q)$. Given that strong double-marginalisation arises as a result of the seller being unable to dictate the terms of trade in the downstream market, these quantities are greater than that which would result from double-marginalisation in the Bertrand price setting game.

If D, T and Y are twice continuously differentiable, and (4.8) and (4.9) are strictly concave, q_s^* and q_m^* are defined implicitly by,

$$D^{-1}(q_s^*) - T^{-1}(q_s^*) - Y^{-1}(q_s^*) = q_s^* \frac{d}{dq_s^*}\big(T^{-1}(q_s^*) - D^{-1}(q_s^*)\big), \tag{4.10}$$

and

$$D^{-1}(q_m^*) - T^{-1}(q_m^*) - Y^{-1}(q_m^*) = q_m^* \frac{d}{dq_m^*}\big(Y^{-1}(q_m^*) - D^{-1}(q_m^*)\big). \tag{4.11}$$

The terms of the RHS of both equations are positive implying that both q_s^* and q_m^* are less than the market clearing quantity.

Proposition 5. *Suppose that x_b satisfies* (3.1) *and that D is twice continuously differentiable and strictly downward sloping. Suppose further that the cost functions of the seller and intermediary satisfy $C_j' \geq 0$ and $C_j'' \geq 0$, and are likewise twice continuously differentiable. Finally, suppose that* (4.8) *and* (4.9) *are strictly concave. For $\hat{p}^d = D^{-1}(\hat{q})$, the allocation $x(\hat{p}, \hat{q})$ is not in* $\text{Core}(N, V)$ *if $\hat{q} < \min\{q_m^*, q_s^*\}$.*

The proof is a straightforward extension of example 4.1. The trick lies in demonstrating that the upstream market price can transfer value between the seller and intermediary over a wide enough range of values that they gain a common interest in increasing the quantity traded when the quantity is less than $\min\{q_m^*, q_s^*\}$.

Proposition 5 establishes that in well behaved cases the quantity traded cannot be less than the quantities that the either the seller or the intermediary would specify if they could dictate the price in both markets, nevertheless it may be the case that vertical integration can deliver efficiency gains. To see this note that if a vertically integrated monopoly were able to dictate a the price for its output it would choose to trade the quantity,

$$q_v^* = \underset{q}{\operatorname{argmax}}\Big(p^d q - \int_0^q \big(Y^{-1}(\vartheta) + T^{-1}(\vartheta)\big)d\vartheta\Big), \tag{4.12}$$

where $p^d = D^{-1}(q)$. If (4.12) is strictly concave q_v^* must satisfy,

$$D^{-1}(q_v^*) - T^{-1}(q_v^*) - Y^{-1}(q_v^*) = -q_v^* \frac{d}{dq_v^*} D^{-1}(q_v^*). \tag{4.13}$$

Comparing (4.13) to (4.10) and (4.11) we see that $q_v^* > q_s^*$ if $T^{-1}(q_s^*) > 0$ and $q_v^* > q_m^*$ if $Y^{-1}(q_m^*) > 0$. In words, the quantity that maximises the vertically integrated monopoly's payoff will be strictly greater, and more efficient, than quantity preferred by either the seller or intermediary, if both C_s and C_m are strictly convex.

This weak double-marginalisation arises because a seller (or intermediary), with the ability to dictate the price in both markets simultaneously, does not take into account the profit that accrues to the intermediary (seller) when the intermediary's markup (seller's price) is set at marginal cost. This profit cannot be transferred between players via the upstream market price and therefore does not enter into the decision of the price setter. Conversely, a merged entity is motivated by total industry profit, resulting in an optimal quantity that is greater than the quantity that a price setting seller (or intermediary) would choose.

5 Discussion

The constrained coalitional price setting game presents a viable alternative to Bertrand price setting in markets for homogeneous and spatially differentiated goods. Its predictions are precise, robust and remarkably well behaved. The results developed in this paper reinforce the intuition of Shy (2001) that an undercut proof-criteria is an appropriate solution concept to apply to spatial models with finite buyers. While in markets for homogeneous goods the constrained coalitional price setting game restores the primacy of market clearing outcomes, even where sellers face strictly convex costs.

At first glance the constrained coalitional price setting game may seem like a radical departure from Bertrand price setting. However, in reality it is nothing more than another means of selecting a stable outcome (or outcomes) from a set of outcomes that are feasible given the technology of the market. The only distinction lies in the way in which we assume that interaction between the players in a market brings about the vector of prevailing prices. Bertrand price setting requires each seller to make a unilateral and irrevocable commitment to a price, while the constrained coalitional approach view prices as arising from unstructured interaction amongst all market participants.

The importance of the constraints imposed on the feasible set by the price mechanism can be clearly seen in the existence of inefficient outcomes within the core. In many cases, such as the multilateral monopoly examples of sections 3 and 4, inefficient outcomes lie in the core because of the inability of players to arbitrarily reassign the value that they create. Moreover, as shown in the discussion of double-marginalisation, this inability to arbitrarily reassign value may lead to a weak form of double-marginalisation, although the distortion cannot be as large as that which arises in Bertrand price competition.

In this final section we consider why one might prefer the constrained coalitional price setting game to Bertrand price setting, compare and contrast the results produced by both models, and discuss some implications for antitrust policy.

5.1 Why Prefer a Coalitional Approach?

Thus far we have said little about why one might prefer constrained coalitional price setting over Bertrand. One factor in favour of the constrained coalitional approach is certainly tractability. The results developed in this paper demonstrate that in a number of important settings the constrained coalitional price setting game is considerably easier to work with than Bertrand price setting, and produces precise results that can be readily tested.

Despite belonging to the NTU family of coalitional games — a type of game that is notoriously difficult to solve given the arbitrary complexity of the feasible sets — the constrained coalitional price setting game is remarkably well behaved. The structure provided by the price constraint allows us to use straightforward conditions such as undercut-proofness (proposition 1) and market clearing (proposition 2) to construct the core.

Yet convenience alone cannot justify the use of a particular modelling technique. We must also be mindful of the assumptions that motivate the structure of the model. The Bertrand price setting game does not perform well in this regard.

When noncooperative game theory is first presented to undergraduate students the motivating example is usually the prisoners' dilemma. The prisoners' dilemma is a wonderfully simple game that engages the attention of students with its counter intuitive solution. More importantly from our point of view, the circumstances of the motivating example precisely match the structure of a one shot simultaneous moves game. Each player in the prisoners' dilemma is locked in a cell, ignorant of the action being taken by the other player. Moreover, the action — to confess or not confess — can only be taken once, and cannot be altered once the player has access to more information.

The same cannot be said of a Bertrand price setting game when the prices being modelled will persist for more than an instant. In many real world situations a seller will observe the prices set by her rivals before the majority of buyers. Given the ease with which most sellers can change their respective prices, such a seller has the opportunity to alter her price before the majority of transactions occur.

The structure of the Bertrand price setting game simply does not match the reality that it is attempting to model. The attraction of Bertrand price setting lies in the ex-post stability of a pure strategy Bertrand Nash equilibrium. Where an equilibrium exists in pure strategies a seller who observes all the prices set by her rivals has no incentive to unilaterally alter her price; all prices are mutual best responses. This is a stability based on complete knowledge rather than mutually correct anticipation. Mixed strategy Bertrand Nash equilibria do not share this property. Once every seller is able to observe the prices that result from her rivals employing their equilibrium strategy mix, at least one seller will have an incentive to unilaterally alter her price.

A one shot noncooperative game is unsatisfactory for modelling prices that will persist for more than an instant when an equilibrium only exists in mixed strategies. For this reason many authors prefer to employ a dynamic structure when addressing problems such as the switching costs model. Yet these models are not without their difficulties.

Whether we are considering a game with repeated simultaneous moves, or repeated sequential moves,[20] dynamic models of price formation rely on the reasonable assumption that sellers can rapidly change their prices in order to produce discount factors that are sufficiently close to one to support the full range of equilibria. The possibility of rapidly changing prices creates two distinct problems for a dynamic Bertrand price setting game: First, the more rapid are the price movements the worse will be the quality of information in the market. In the limit no buyer will be able to observe more than one price at a time and each seller becomes a local monopolist relative to the demand of those buyers observing only her price in each period. The second difficulty arises on the demand side. Over a short enough time horizon almost all consumption can either be delayed or stored. This gives buyers an ability to act strategically in a manner that could have a substantial impact on the payoffs to the various price choices. Buyers could provide disproportionate rewards to sellers posting a low price in a particular period, and conversely punish sellers who set high prices. This contrast with the standard models in which sellers face a static demand in each period.

Limits on information and the strategic options of buyers could, of course, be incorporated into the structure of a noncooperative game. However, imposing more structure upon a noncooperative game does tend to reduce the range of real world situation to which the model may be applied. Moreover, the structure itself may be a factor that can be manipulated by the players in a market.

As is the case in models of bargaining, the alternative to adding more

[20]See for example Maskin and Tirole (1988).

structure is to eliminate structure entirely. The set of feasible outcomes in an unstructured game must be constrained to account for the "technology" employed in transactions — in this case the price mechanism — but within these bounds we allow any player to propose any outcome. The core then represents the set of outcomes that are stable. This stability always persists ex-post. By definition when the prevailing prices implement a core allocation no coalition of players can agree a deviation in which they would all (weakly) benefit.

Ex-post stability allows the constrained coalitional price setting game to model prices that are expected to persist for some period of time. For so long as the identity of players and the form of their payoffs do not change, core outcomes remain ex-post stable and no coalition has an incentive to deviate.

Moreover, the constrained coalitional prices setting game does not disenfranchise buyers. Buyers take an active role in the game. This allows constrained coalitional price setting to be applied to markets with few buyers, such as intermediate markets, where we would expect to see buyers playing a role in determining the terms of trade.

The constrained coalitional approach to price setting is not without its difficulties. Throughout this paper we have glibly assumed that when a deviating coalition forms it leaves "the market" and operates entirely independently. This assumption is innocuous in the context of this paper as by assumption each player's payoff depends only on the trades to which he is a party

However, where consumption is associated with externalities or network effects disjoint coalitions of buyers and sellers will not necessarily be able to operate independently. A decision by a coalition to separate itself from the remainder of the market need not also sever the links through which the externalities flow. It follows that in the presence of externalities or network effects, the viability of a deviation will depend both upon the terms of trade that can be negotiated within the deviating coalition, and any externalities that will be produced by the remaining players.

Determining how the actions of the remaining players affect the payoffs of a deviating coalition is a common problem in coalitional game theory.

Should we assume that the remaining players will act to minimise the payoffs of deviating players, or will they seek to maximise their own payoffs? To be consistent with the structure of the constrained coalitional price setting game we would require the choice between adopting an aggressive or defensive stance to be made with the choice of price, retaining the assumption that q must satisfy rational rationing.

It is also worth considering how the members of a deviating coalition might come together. In an end product market in which the buyers are consumers we would imagine that sellers would take a lead role in assembling a deviating coalition. Sellers could use direct mail and loyalty schemes to target a particular subset of buyers with whom they would like to defect. Other deviations, such as barter systems evolve in a more organic way, sometimes with the aid of a third party.

Pulling together a deviating coalition is not without its costs. For this reason it may be more appropriate to use the ε-core to define the set of stable market outcomes. The ε-core only permits an allocation to be blocked if there exists a deviation that benefits each participating player by an amount that is at least equal to ε. In the NTU framework of the constrained coalitional price setting game we could attribute different base ε to each player acknowledging the asymmetric role that they play in bringing a coalition together. Nevertheless, the core always lies within the ε-core, and can be regarded as containing the set of super-stable outcomes.

5.2 Unstructured Interaction

It is worth taking a moment to consider what is meant by an unstructured interaction. When referring to a bargaining problem — possibly the most common and least contentious application of coalitional game theory — players interact by issuing offers, counter offers and threats amongst other things. This interpretation cannot generally be extended to the interaction between buyers and sellers in an end product market.

Nevertheless one can view the prices in a market as being the product of an "implicit" bargaining process. There are many ways in which buyers and sellers can communicate with one and other. Consider, for example, a period of time in which sellers makes successive offers to the market by posting prices, and buyers indicate their acceptance of these terms through the quantities that they purchase. In this way we can view periods of price instability in a market as a process of negotiation over prices, and the subsequent period of price stability as the outcome of these negotiations. Given the ex-post stability of a core outcome, we would only expect price instability when one of the market fundamentals changes.

Buyers always have the option to use their consumption to send a message to sellers, employing the quantities they purchase to reward or punish a seller. The most extreme example of this behaviour is the consumer boycott. While often employed to bring about a change to the quality of a seller's product, a consumer boycott could equally be invoked as an extreme sanction in a negotiation over price. This is particularly true where a seller is perceived to have acted egregiously; for example by price gouging.

Buyers have other means of communicating with and punishing sellers. A topical example is consumer reaction to the increasing price of petrol in recent years. Representatives of buyers in this market — motoring groups and politicians — have issued warnings against profiteering and called for anti-trust authorities to monitor retail prices closely. In extreme cases these calls have resulted in enquiries that are costly to the sellers involved, and damaging to their public image. Similar pressures have been placed on insurance companies following major disasters. In most cases the sellers are not acting illegally, however vocal representatives clearly state the bargaining position of those buyers active in the market.

5.3 Implication for Antitrust Regulations

In many jurisdictions simply discussing pricing with a rival is considered to be *per se* anti-competitive. The structure of the Bertrand price setting game has reinforced this view. Regulators reason that Bertrand outcomes tend to be more efficient than monopoly outcomes. Efficiency is therefore served if sellers are prevented from colluding to set these monopoly outcomes.

The results generated by the constrained coalitional price setting game tend to contradict this view. The unstructured interaction that produces prices in this game permits unlimited communication between sellers and specifically allow seller to co-ordinate their prices. Collusion is prevented because buyers are also active players in this interaction.

But we can go further still. The following example demonstrates that even if sellers are permitted to make side payments, sellers will still not be able to implement a collusive outcome in the constrained coalitional price setting game.

Example 5.1. Consider a market for a homogeneous good with two buyers b_1 and b_2, and two sellers s_1 and s_2. Suppose that the sellers have unlimited capacity at a constant marginal cost of zero, and that each buyer faces the demand function $D_i(\rho) = (1-\rho)/2$.

This example satisfies the assumptions of the generalised spatial model developed in section 2 and therefore we may employ undercut-proofness to find the solution to a baseline case in which sellers cannot make side payments. Specifically, the core contains a single allocation that delivers each buyer a payoff of 1/4 while both sellers receive a payoff of zero.[21]

Now suppose that sellers can make side payments to one and other. The form of each buyer's payoff remains,

$$x_{b_1}(p,q) = x_{b_2}(p,q) = \frac{1}{4}\big(1 - \min\{p_1, p_2\}\big)^2, \tag{5.1}$$

while the payoffs to the two sellers must satisfy,

$$x_{s_1} + x_{s_2} = \min\{p_1, p_2\}\big(1 - \min\{p_1, p_2\}\big), \tag{5.2}$$

which is too say that sellers are constrained only insofar as the sum of seller payoffs must equal the total surplus that accrues to that side of the market. Importantly, sellers are now free to pay one and other in return for setting a high price.

Despite this additional freedom sellers remain unable extract rents from this market. To see this suppose that $\min\{p_1, p_2\} > 0$ and without loss of

[21]These payoffs can only be implemented by a price vector with at least one zero entry.

generality suppose that $x_{s_1} \leq x_{s_2}$. By (5.2) the payoff to s_1 must satisfy,

$$x_{s_1} \leq \frac{1}{2} \min\{p_1, p_2\}\big(1 - \min\{p_1, p_2\}\big). \tag{5.3}$$

We wish to show that this outcome cannot arise in the core. Consider a deviation of the coalition $G = \{b_1, b_2, s_1\}$ agreeing to the price $p_1 = \frac{1}{2} \min\{p_1, p_2\}$. This deviation strictly increases the payoff to both the buyers and seller 1. ■

While side payments do provide sellers with a common incentive to maximise industry profits, they fail to facilitate collusion because the sellers remain in conflict over how to divide the surplus that accrues to their side of the market. This result readily generalises and demonstrates that even if the sellers are allowed to pay one and other to participate in price fixing, the market clearing outcome will always remain in the core. Indeed in many cases, such as the one outlined above, the market clearing outcome remains the only outcome in the core.

So how then can collusion be facilitated? One obvious answer is through a market sharing agreement. If the sellers can merge in advance of the market each seller gets a fixed share of industry income, thereby removing the contest for this income from the price sharing game.[22] Alternatively, and again in advance of the market, the sellers could agree to divide the market. In the example above seller 1 could commit not to trade with buyer 2 and vice versa. This would leave each seller as a local monopolist. In either case the collusive agreement would only hold if the share determined in the preliminary stage is binding.

Far from facilitating collusion, communication in the constrained coalitional price setting game tends to generate efficient outcomes and certainly performs better in this regard than does the Bertrand price setting game. If it is truly the case that buyers are completely passive participants in a market — as is assumed in the Bertrand model — the results of this paper

[22] For $\alpha_1 \in (0, 1)$ and $\alpha_2 = 1 - \alpha_1$ a merger or joint venture would cause seller payoffs to become,

$$x_{s_j} = \alpha_j \min\{p_1, p_2\}\big(1 - \min\{p_1, p_2\}\big),$$

where α_j is seller j's share of the joint venture.

suggest that a considerable social gain could be effected by encouraging buyers to take a more active role in the determination of prices. However, if price formation has more in common with the unstructured interaction assumed by the constrained coalitional price setting game then we must reconsider the nature of collusion and the practices that facilitate it.

5.4 Concluding Remarks

That a coalitional approach should find a pure strategy solution where the Bertrand price setting game failed, was anticipated by Edgeworth (1925). Edgeworth describes an equilibrium market outcome as being,

> "...defined by the condition that no individual in any group, whether of buyers or sellers, can make a new contract with individuals in the other groups, such that all the re-contracting parties should be better off than they were under the preceding system of contracts."

This is one of the earliest references to a core like concept of stability.

In many places the predictions of Bertrand and constrained coalitional price setting are in direct conflict. Two prominent cases can be found in this paper: Kreps and Scheinkman (1983) demonstrated that where sellers have capacities that are greater than Cournot best responses, the support of the mixed strategy Bertrand Nash equilibrium may be bounded away from the market clearing price. In contrast proposition 3 suggest that there is a sense in which the market clearing price is the only viable outcome when competition exists on both sides of the market.

The intuition behind this result is straightforward. Buyers are completely passive in the Bertrand price setting game, unable to exercise any strategic option during the determination of prices. Sellers are able to unilaterally increase their price and take for granted that buyers will follow them. For this reason a seller's payoff in the mixed strategy Bertrand Nash equilibrium must be at least as great as the payoff that the seller can achieve by setting the monopoly price relative to residual demand. In the Constrained coalitional price setting game all players are involved in the process of price formation.

Sellers cannot benefit as a result of a deviation unless they can convince buyers to come with them. As a buyer will not agree to a price rise unless it is accompanied by an increase in the quantity allocated to the buyer, a deviation to a monopoly price will not generally be possible for a seller.

The second notable point of conflict concerns the possibility that double-marginalisation may arise in vertically related markets. Double-marginalisation cannot be a core outcome of the constrained coalitional price setting game because the allocation can be blocked by a co-ordinated price movement that benefits all players. We would expect similar results in parallel markets for complementary goods.

In both examples considered here the core outcomes are more efficient than the corresponding Bertrand Nash equilibria. In the case of convex costs this is because the active participation of buyers acts as a limit on the distortions that sellers can impose on the market. In the case of double-marginalisation increased efficiency results from the possibility of co-operation and co-ordination.

The ease with which the results of this paper are developed is due in large part to the assumption that products are perfect substitutes (up to the limit of a spatial characteristic), and that each buyer's payoff is independent of the consumption of other buyers. That being said, the constrained coalitional price setting game does produce precise results in cases where the Bertrand price setting game is at best ambiguous in its predictions.

If the constrained coalitional price setting game is to emerge as a viable alternative to Bertrand price setting this ease of use must extend to more complex settings. The game is yet to be tested where the products being produced by different sellers are compliments or partial substitutes. Various forms of externalities, including network externalities, likewise warrant investigation.

A Proof of Lemma 2

The integrals on either side of the equality are identical by construction. If $q_i = (0, \ldots, 0, q_{im}, 0, \ldots, 0)$, (2.5) holds trivially. We are left with the case in which q_i has at least two strictly positive entries.

Let q_{ij} and q_{ik} be any two strictly positive entries in q_i, it must be the case that $p_j + k_{ij} = p_k + k_{ik}$. Suppose to the contrary that $p_j + k_{ij} > p_k + k_{ik}$. But b_i could increase x_i by reducing q_{ij} to zero and increasing q_{ik} to $q_{ij} + q_{ik}$, thereby lowering the effective unit price paid by b_i and contradicting the assumption that q_i maximises x_i.

Again letting q_{ij} and q_{ik} be any two strictly positive entries in q_i, it must also be the case that $K_{ij} = K_{ik} = 0$. Suppose to the contrary that $K_{ij} > 0$, and without loss of generality that $K_{ij} \geq K_{ik}$. But b_i could increase x_i by reducing q_{ij} to zero and increasing q_{ik} to $q_{ij} + q_{ik}$, thereby foregoing the need for b_i to incur the fixed cost K_{ij} and contradicting the assumption that q_i maximises x_i.

It follows that if q_i has more than one strictly positive entry, for any $s_m \in S \cap G$ such that $q_{im} > 0$, (2.5) can be satisfied by setting $\hat{q}_{im} = \sum_{j \in S} q_{ij}$. Moreover, the transactions $\hat{q}_i = (0, \ldots, 0, \hat{q}_{im}, 0, \ldots, 0)$ maximise x_i given p and G, and therefore there exists a $\hat{q} \in Q^r(p^*, q^*)$ with $\hat{q}_i$ as its ith component.

B Proof of Proposition 1

i. follows directly from the individual rationality of the sellers.

For *ii.* the *only if* part is straightforward. Suppose to the contrary that there exists $\hat{p}_j < p^*_j$ such that,

$$x_j(p^*, q^*) \leq (\hat{p}_j - c_j) \sum_{i \in B} q_{ij}. \tag{B.1}$$

for some $q \in Q^r\big((p^*_{-j}, \hat{p}_j), N\big)$. Define $B_j(q) = \{b_i \in B : q_{ij} > 0\}$. It follows from lemma 2 that the coalition $s_j \cup B_j(q)$ could deviate, agreeing to the price $\hat{p}_j$, without leaving any player in the coalition worse off. To complete the contradiction we need only show that at least one member of the coalition will be strictly better off as a result of the deviation.

If there exists $b_i \in B_j(q)$ such that $q^*_{ij} > 0$, b_i strictly benefits as a result of paying a lower price. We are left with the case in which $q^*_{ij} = 0$ for all $b_i \in B_j(q)$ and as a consequence $x_j(p^*, q^*) = 0$. By assumption there exists a

$c_j < \bar{p}_j < p_j^*$ and $q \in Q^r\big((p_{-j}^*, \bar{p}_j), N\big)$ such that $q_{ij} > 0$, thus the allocation $x\big((p_{-j}^*, \bar{p}_j), q\big)$ with $x_j > 0$ is feasible in (N, V).

Now suppose that a coalition $G \subseteq N$ agreeing to a price vector $\hat{p}$ and transactions $\hat{q} \in Q^r(\hat{p}, G)$, can block the allocation $x(p^*, q^*)$. To prove the *if* part of *ii.* we need only show that there exists a seller $s_j \in G$, and $\tilde{q} \in Q^r\left((p_{-j}^*, \hat{p}_j), N\right)$ such that the coalition $s_j \cup B_j(\tilde{q})$ agreeing to the pair $(\hat{p}_j, \tilde{q})$, can likewise block $x(p^*, q^*)$.

First, and trivially, we note that G cannot consist entirely of either buyers or sellers. Rational rationing ensures that both buyers an sellers receive non-negative payoffs. A coalition consisting entirely of either buyers or sellers will generate a payoff of zero for each of its members and as such no player can strictly benefit.

Next we show that if there exists $s_j \in G$ such that $\hat{p}_j > p_j^*$, it must be the case that s_j makes no sales and receives a payoff of zero as a result of the deviation. Suppose to the contrary that a buyer $b_i \in G$ purchases q_{ij} units from s_j as a result of the deviation. But b_i can purchase from s_j when the prevailing price vector is p^* implying that the deviation must leave b_i strictly worse off. It follows that s_j makes no sales and receives no benefit from the deviation. Moreover, it must be the case that the coalition $G \setminus s_j$, agreeing to the price vector $\hat{p}$, is also capable of blocking the allocation.

Finally we show that if the coalition G can block an allocation $x(p^*, q^*)$, there exists a seller $s_j \in G$ such that the coalition $s_j \cup B_j(\tilde{q})$, agreeing to the price $\hat{p}_j$, can likewise block the allocation. We can confine our attention to deviating coalitions in which $\hat{p}_j \leq p_j^*$ for all $s_j \in G$.

Given that $x_i(\hat{p}, \hat{q}) \geq x_i(p^*, q^*)$ for all $b_i \in G$, for all $s_j \in G$ there must exists $\tilde{q} \in Q^r\big((p_{-j}^*, \hat{p}_j), N\big)$ such that $B_j(\hat{q}) \subseteq B_j(\tilde{q})$. In other words, if a buyer in G is willing to buy from s_j where the prevailing prices are $\hat{p}$, he must be willing to buy from s_j where s_j retains the lower price $\hat{p}_j$ but every other seller sets their respective price from the vector p^*. Moreover the trades $\tilde{q}$ grant s_j a weakly greater volume of sales than $\hat{q}$, thereby weakly increasing s_j's payoff.

One player in G must strictly benefit from the deviation. If that player is the seller s_j, it follows from lemma 2 that the coalition $s_j \cup B_j(\tilde{q})$, agreeing to the pair $(\hat{p}_j, \tilde{q})$, can block the allocation $x(p^*, q^*)$. If the player that strictly benefits is the buyer b_i, pick a seller $s_j \in G$ such that $\hat{q}_{ij} > 0$, and once again by lemma 2 the coalition $s_j \cup B_j(\tilde{q})$, agreeing to the pair $(\hat{p}_j, \tilde{q})$, can block the allocation $x(p^*, q^*)$.

C Proof of Proposition 2

From the aggregate measures of demand and cost it is possible to construct a function $w : 2^N \longrightarrow \mathbb{R}$, such that $w(G) = 0$ for any coalition that does not contain at least one buyer and one seller, and,

$$w(G) = \int_0^{D_G(p_G^c)} \big(D_G^{-1}(\vartheta) - Y_G^{-1}(\vartheta)\big) d\vartheta, \tag{C.1}$$

otherwise. Notice that $w(G)$ is the greatest total value that can be generated by trades between the members of the coalition G. It follows that (N, V) and w satisfy the relationship set out in (1.5).

It follows from lemma 1 that in order to prove proposition 2 it is only necessary to show that the allocation $x(p^*, q^*)$ lies in the core of (N, w). We proceed via a series of lemmas.

Definition 4. Let x be an allocation in the game (N, w). The *effective unit price* $\bar{p}_i$, paid by a buyer $b_i \in G$ is defined as,

$$\bar{p}_i = \frac{\int_0^{D_i(p_G^c)} D_i^{-1}(\vartheta) d\vartheta - x_i}{D_i(p_G^c)}. \tag{C.2}$$

Similarly, the *effective unit price* $\bar{p}_j$, received by a seller $s_j \in G$ is defined as,

$$\bar{p}_j = \frac{x_j + \int_0^{Y_j(p_G^c)} Y_j^{-1}(\vartheta) d\vartheta}{Y_j(p_G^c)}. \tag{C.3}$$

For a coalition G, $\bar{p}_i$ is uniquely defined by x_i and vice versa. Moreover given that,

$$\sum_{i \in B \cap G} D_i(p_G^c) = D_G(p_G^c) = Y_G(p_G^c) = \sum_{j \in S \cap G} Y_j(p_G^c) \tag{C.4}$$

we have,

$$\frac{\sum_{i \in B \cap G} D_i(p_G^c)\bar{p}_i}{D_G(p_G^c)} = \frac{\sum_{j \in S \cap G} Y_j(p_G^c)\bar{p}_j}{Y_G(p_G^c)}. \tag{C.5}$$

In words the weighted average of effective unit prices paid by buyers is equal to the weighted average of effective unit prices received by sellers.

Lemma 3. *Suppose that buyer payoffs satisfy* (3.1). *For all* $b_i \in B$ *and* $q \in \mathbb{R}_+$,

$$\int_0^{D_i(p_N^c)} D_i^{-1}(\vartheta)d\vartheta - D_i(p_N^c)p_N^c < \int_0^q D_i^{-1}(\vartheta)d\vartheta - q\bar{p}_i, \tag{C.6}$$

implies $\bar{p}_i < p_N^c$.

Proof. The expression on the RHS of (C.6) is (weakly) concave in q. When $\bar{p}_i = p_N^c$ the RHS is maximised where $q = D_i(p_N^c)$. It follows that in order to satisfy the inequality we must have $\bar{p}_i < p_N^c$. □

Lemma 4. *Suppose that seller cost functions satisfy* $C_j' \geq 0$ *and* $C_j'' \geq 0$. *For all* $s_j \in S$ *and* $q \in \mathbb{R}_+$,

$$Y_j(p_N^c)p_N^c - \int_0^{Y_j(p_N^c)} Y_j^{-1}(\vartheta)d\vartheta < q\bar{p}_j - \int_0^q Y_j^{-1}(\vartheta)d\vartheta, \tag{C.7}$$

implies $\bar{p}_j > p_N^c$.

Proof. Follows directly from the proof of lemma 3. □

Lemma 5. *Suppose that buyer payoffs satisfy* (3.1), *and that seller cost functions satisfy* $C_j' \geq 0$ *and* $C_j'' \geq 0$. *The allocation* x^* *is in* $\text{Core}(N, w)$ *if the corresponding effective unit price* $\bar{p}_i^* = p_N^c$ *for all* $i \in N$.

Proof. Buy way of contradiction suppose that there exists a coalition G such that,

$$\sum_{i \in G} x_i^* < w(G). \tag{C.8}$$

Let x be an allocation satisfying $\sum_{i \in G} x_i = w(G)$ and $x_i > x_i^*$ for all $i \in G$. Moreover define $\bar{p}_i$ as the effective unit price corresponding to the payoff x_i.

From lemma 3 we know that $\bar{p}_i < \bar{p}_i^* = p_N^c$ for all $i \in B \cap G$, while from lemma 4 we know that $\bar{p}_j > \bar{p}_j^* = p_N^c$ for all $i \in S \cap G$. This implies,

$$\frac{\sum_{i \in B \cap G} D_i(p_G^c)\bar{p}_i}{D_G(p_G^c)} < p_N^c < \frac{\sum_{j \in S \cap G} Y_j(p_G^c)\bar{p}_j}{Y_j(p_G^c)}, \tag{C.9}$$

contradicting (C.5). □

D Proof of Proposition 3

The proof proceeds via a series of lemmas. Lemma 6 presents an existence result that is employed by lemma 8 to prove that if p^* induces an allocation that lies in the core of (N_n, V) for all $n \in \mathbb{N}$ then all trading sellers must set the same price. Lemma 9 illustrates that if p^* induces an allocation that lies in the core of (N_n, V) for all $n \in \mathbb{N}$ then p^* cannot leave any buyer with excess demand, similarly lemma 10 demonstrates that p^* cannot leave any seller with excess capacity.

Lemma 6. *For all $\varepsilon > 0$ and $\delta \in (0,1)$ we can select $m, n \in \mathbb{N}$ with $n \geq \underline{n} \in \mathbb{N}$ such that $\frac{1}{n}\varepsilon > \frac{m}{n} - \delta \geq 0$.*

Proof. If δ is rational then set $\frac{m}{n} = \delta$ and we are done. We are left with the case in which δ is not rational. For all $\underline{n} \in \mathbb{N}$ we can define an m_0 such that,

$$\frac{m_0}{\underline{n}} > \delta > \frac{m_0 - 1}{\underline{n}}. \tag{D.1}$$

Now define α_i such that,

$$\alpha_i = \min\left\{\beta \in \mathbb{N} : \frac{\beta m_{i-1} - 1}{\beta \prod_{\phi=1}^{i-1} \alpha_\phi \underline{n}} > \delta\right\}, \tag{D.2}$$

for all $i \in \mathbb{N}$, where $m_i = \alpha_i m_{i-1} - 1$.[23] From the construction of α_i we can see that,

$$\frac{\alpha_i m_{i-1} - 1}{\prod_{\phi=1}^{i} \alpha_\phi \underline{n}} > \delta > \frac{(\alpha_i - 1)m_{i-1} - 1}{\prod_{\phi=1}^{i-1} \alpha_\phi (\alpha_i - 1)\underline{n}}, \tag{D.3}$$

[23]We follow the convention that $\prod_{\phi=1}^{0} \alpha_\phi = 1$.

which in turn implies,

$$\frac{m_i}{\prod_{\phi=1}^{i}\alpha_\phi \underline{n}} - \delta < \frac{1}{\prod_{\phi=1}^{i}\alpha_\phi(\alpha_i - 1)\underline{n}}. \tag{D.4}$$

To complete the proof we need only show that there exists an $i \in \mathbb{N}$ such that if we set $n = \prod_{\phi=1}^{i}\alpha_\phi \underline{n}$ and $m = m_i$ then,

$$\frac{m}{n} - \delta < \frac{1}{n(\alpha_i - 1)} < \frac{\varepsilon}{n}. \tag{D.5}$$

Combining (D.2) and (D.4) we see that for all $i \in \mathbb{N}$, it must be the case that $\alpha_{i+1} \geq \alpha_i$. It is sufficient to show that for all i there must exist an $i' > i$ such that $\alpha_{i'} > \alpha_i$ and as a consequence for all $\frac{1}{\varepsilon} \in \mathbb{R}_+$ there exists an $i \in \mathbb{N}$ such that $\alpha_i - 1 > \frac{1}{\varepsilon}$.

Suppose to the contrary that there exists an i^* such that $\alpha_i = \alpha_{i^*}$ for all $i > i^*$. The LHS and RHS of (D.3) can now be written as,

$$\frac{\alpha_i m_{i-1} - 1}{\prod_{\phi=1}^{i}\alpha_\phi \underline{n}} = \frac{1}{\underline{n}}\left(m_0 - \sum_{\phi=1}^{i^*-1}\frac{1}{\prod_{\theta=1}^{\phi}\alpha_\theta} - \frac{1}{\prod_{\theta=1}^{i^*-1}\alpha_\theta}\sum_{\phi=i^*}^{i}\frac{1}{\alpha_{i^*}^{\phi-i^*+1}}\right), \tag{D.6}$$

and

$$\begin{aligned}\frac{(\alpha_i - 1)m_{i-1} - 1}{\prod_{\phi=1}^{i-1}\alpha_\phi(\alpha_i - 1)\underline{n}} = \frac{1}{\underline{n}}\Bigg[m_0 - \sum_{\phi=1}^{i^*-1}\frac{1}{\prod_{\theta=1}^{\phi}\alpha_\theta}\\ - \frac{1}{\prod_{\theta=1}^{i^*-1}\alpha_\theta}\left(\sum_{\phi=i^*}^{i-1}\frac{1}{\alpha_{i^*}^{\phi-i^*+1}} + \frac{1}{(\alpha_{i^*} - 1)\alpha_{i^*}^{i-i^*}}\right)\Bigg],\end{aligned} \tag{D.7}$$

respectively. The limit of both (D.6) and (D.7) is,

$$\begin{aligned}\lim_{i\to\infty}\frac{\alpha_i m_{i-1} - 1}{\prod_{\phi=1}^{i}\alpha_\phi \underline{n}} &= \lim_{i\to\infty}\frac{(\alpha_i - 1)m_{i-1} - 1}{\prod_{\phi=1}^{i-1}\alpha_\phi(\alpha_i - 1)\underline{n}}\\ &= \frac{1}{\underline{n}}\left(m_0 - \sum_{\phi=1}^{i^*-1}\frac{1}{\prod_{\theta=1}^{\phi}\alpha_\theta} - \frac{1}{(\alpha_{i^*} - 1)\prod_{\theta=1}^{i^*-1}\alpha_\theta}\right).\end{aligned} \tag{D.8}$$

The term on the RHS of (D.8) must be equal to δ given that it is the limit of δ's upper and lower bounds, and is unambiguously a rational number: A contradiction. □

Lemma 7. *Suppose that buyer payoffs satisfy* (3.1) *with* D_i *twice continuously differentiable and strictly downward sloping for all* $i \in B$. *Suppose further that* $|S| \geq 2$ *and that* $C_j' \geq 0$, $C_j'' \geq 0$, $Y_j(p_N^c) > 0$ *and* $p_N^c \neq p_{N-j}^c$ *for all* $j \in S$.

Relabelling if necessary let $j < k$ *imply* $p_j^* \leq p_k^*$, *and assume that* $Y_j(p_j^*) > 0$ *for all* $j \in S$.[24] *Suppose that for all* $n \in \mathbb{N}$ *there exists a* $q^* \in Q^r(p^*, N_n)$ *such that* $x(p^*, q^*)$ *lies in the core of* (N_n, V). *It must be the case that for all* $j \in S$ *we can find an* $\underline{n} \in \mathbb{N}$ *such that for all* $n \geq \underline{n}$,

$$\sum_{\substack{i \in B \\ k \in \{1,\dots,n\}}} q_{ikj}^* > 0. \tag{D.9}$$

Proof. Suppose to the contrary that there exists $\hat{j} \in S$ such that for all $\underline{n} \in \mathbb{N}$ there exists an $n \geq \underline{n}$ for which (D.9) holds with equality. Set $\underline{n}$ such that $D_i(p_N^c)/\underline{n} < Y_{\hat{j}}(p_N^c)$ for all $b_i \in B$, and select $n \geq \underline{n}$ and (p^*, q^*) such that $x(p^*, q^*) \in \text{Core}(N, V)$ and $\sum_{ik \in B^n} q_{ik\hat{j}}^* = 0$. Note that $x_{\hat{j}}(p^*, q^*) = 0$. There are two cases to consider: $p_1^* \leq p_N^c$ and $p_1^* > p_N^c$.

Case 1: $p_1^* > p_N^c$. Select an $i \in B$ such that $0 < D_i(p_N^c)$. The coalition $\{b_{i1}, s_{\hat{j}}\}$ can agree to trade the quantity $\frac{1}{n} D_i(p_N^c)$ at price p_N^c, thereby leaving $s_{\hat{j}}$ no worse off and strictly improving the payoff of b_{i1}: A contradiction.

Case 2: $p_1^* \leq p_N^c$. By assumption we can find a buyer, whom we will designate b_{11} relabelling if necessary, who either experiences excess demand or purchases some quantity at a price that is greater than market clearing. For this buyer,

$$\sum_{j \in S} q_{11j}^* < D_{11}(p_N^c). \tag{D.10}$$

Define $S' = \{j \in S : p_j^* \leq p_N^c\}$ and for all $b_{ik} \in B^n$ let,

$$\bar{q}_{ik} = \max\left\{0, \frac{1}{\underline{n}} D_i(p_N^c) - \sum_{j \in S'} q_{ikj}^*\right\}. \tag{D.11}$$

The scalar $\bar{q}_{ik}$ represents that portion of a buyer's demand that is not satisfied by sellers setting a price no greater than market clearing. Finally, find a set

[24] This assumption is made without loss of generality as a seller s_j can guarantee herself a payoff of zero by setting $p_j^* = \infty$.

of buyers $B' \in B^n$ such that $b_{11} \in B'$ and,

$$\sum_{j \in S'} Y_j(p_j^*) < \sum_{ik \in B'} \Big(\bar{q}_{ik} + \sum_{j \in S'} q_{ikj}^* \Big) \leq Y_{\hat{j}}(p_N^c) + \sum_{j \in S'} Y_j(p_j^*). \tag{D.12}$$

The choice of n ensures that at least one such B' must exist.

Consider a deviation by the coalition $G = B' \cup S' \cup \{\hat{j}\}$, agreeing to the prices $\hat{p}_k = p_k^*$ for all $k \in S'$ and $\hat{p}_{\hat{j}} = p_N^c$, and the trades,

$$\hat{q}_{ikj} = q_{ikj}^* + \frac{\bar{q}_{ik}}{\sum_{\phi \in B'} \bar{q}_\phi} \Big(Y_j(p_j^*) - \sum_{\phi \in B'} q_\phi^* \Big), \tag{D.13}$$

for all $ik \in B'$ and $j \in S'$, and,

$$\hat{q}_{ik\hat{j}} = \max\Big\{ 0, \frac{1}{n} D_i(p_N^c) - \sum_{j \in S'} \hat{q}_{ikj} \Big\}, \tag{D.14}$$

for all $ik \in B'$. This deviation leaves no player worse off and strictly increases the payoff to b_{11}: A contradiction. □

Lemma 8. *Suppose that buyer payoffs satisfy* (3.1) *with D_i twice continuously differentiable and strictly downward sloping for all $i \in B$. Suppose further that $|S| \geq 2$ and that $C_j' \geq 0$, $C_j'' \geq 0$, $Y_j(p_N^c) > 0$ and $p_N^c \neq p_{N-j}^c$ for all $j \in S$.*

Relabelling if necessary let $j < k$ imply $p_j^* \leq p_k^*$. If $p_1^* \neq p_{|S|}^*$, then there exists at least one $n \in \mathbb{N}$ such that,

$$x(p^*, q^*) \notin \text{Core}(N_n, V), \tag{D.15}$$

for all $q^* \in Q^r(p^*, N_n)$.

Proof. Suppose to the contrary that $p_1^* \neq p_{|S|}^*$ and that for all $n \in \mathbb{N}$ there exists $q^* \in Q^r(p^*, N_n)$ such that $x(p^*, q^*)$ lies in the core of (N_n, V). By lemma 7 if $x(p^*, q^*)$ lies in the core of (N_n, V) for all $n \geq \underline{n}$ we must have,

$$\sum_{\substack{i \in B \\ k \in \{1, \dots, n\}}} q_{ik|S|}^* > 0. \tag{D.16}$$

Given that the seller $s_{|S|}$ trades a positive quantity, rational rationing implies that there must exist at least one buyer, whom we will designate b_{11}

relabelling if necessary, for whom $\sum_{j\in S} q_{11j} \leq D_{11}(p^*_{|S|}) < D_{11}(p^*_1)$. Moreover, given that $p^*_{|S|} > p^*_1$, rational rationing implies that $D_N(p^*_1) > Y_1(p^*_1)$ as no buyer would purchase the product at the higher price if excess capacity remained at the lower price.

Let $\hat{i} = \text{argmin}_{i\in B} D_i(p^*_1) - D_i(p^*_{|S|})$. From lemma 6 we know that we can select $m, n \in \mathbb{N}$, with $n \geq \underline{n}$ such that,

$$\frac{m}{n} D_N(p^*_1) \geq Y_1(p^*_1) > \frac{m}{n} D_N(p^*_1) - \frac{1}{n}\big(D_{\hat{i}}(p^*_1) - D_{\hat{i}}(p^*_{|S|})\big)$$
$$\geq \frac{m}{n} D_N(p^*_1) - \frac{1}{n}\big(D_1(p^*_1) - D_1(p^*_{|S|})\big), \quad \text{(D.17)}$$

and note that $m \leq n$. Suppose that the coalition $G = s_1 \cup \{b_{i1}, \ldots, b_{im}\}_{i\in B}$ deviated, agreeing to the price $p_1 = p^*_1$ and quantities $q_{ik} = D_{ik}(p^*_1)$ for all $b_{ik} \in G \setminus b_{11}$ and,

$$q_{11} = \frac{1}{n} D_1(p^*_1) + Y_1(p^*_1) - \frac{m}{n} D_{N_n}(p^*_1) > \frac{1}{n} D_1(p^*_{|S|}). \quad \text{(D.18)}$$

This deviation satisfies the demand of every member of $\{b_{i1}, \ldots, b_{im}\}_{i\in B} \setminus b_{11}$ at the lowest available price while strictly increasing the quantity assigned to b_{11}. Moreover the capacity of s_1 is exhausted at her original price. It follows that every member of $G \setminus b_{11}$ is at least as well off and b_{11}'s payoff strictly increases: A contradiction. □

Lemma 9. *Suppose that buyer payoffs satisfy* (3.1) *with D_i twice continuously differentiable and strictly downward sloping for all $i \in B$. Suppose further that $|S| \geq 2$ and that $C'_j \geq 0$, $C''_j \geq 0$, $Y_j(p^c_N) > 0$ and $p^c_N \neq p^c_{N-j}$ for all $j \in S$.*

Let $p^*_j = \hat{p}^*$ for all $j \in S$. If $Y_N(\hat{p}^*) < D_N(\hat{p}^*)$ then there exists $n \in \mathbb{N}$ such that for all $q^* \in Q^r(p^*, N_n)$, $x(p^*, q^*) \notin \text{Core}(N_n, V)$.

Proof. The proof proceeds in three parts. First we prove that if $x(p^*, q^*)$ lies in the core of (N_n, V) and there exists i, k and j such that $q^*_{ikj} > 0$ then $\sum_{j\in S} q^*_{ikj} > 0$ for all $k \in \{1, \ldots, n\}$. Next we prove that for all $i \in B$ if $x(p^*, q^*)$ lies in the core of (N_n, V) and there exists k and j such that $q^*_{ikj} > 0$, then,

$$\sum_{\substack{j\in S \\ k\in\{1,\ldots,n\}}} q^*_{ikj} > \frac{n-1}{n} D_i(\hat{p}^*). \quad \text{(D.19)}$$

Finally we prove that for sufficiently large $n \in \mathbb{N}$, if there exists $q^* \in Q^r(p^*, N_n)$ such that $x(p^*, q^*)$ lies in the core of (N_n, V), then for all $i \in B$ such that $D_i(\hat{p}^*) > 0$,

$$\sum_{\substack{j \in S \\ k \in \{1, \cdots, n\}}} q^*_{ikj} > 0. \tag{D.20}$$

Step 1. Suppose to the contrary that there exists an $r \in B$ and $n_1, n_2 \in \{1, \ldots, n\}$ such that $q^*_{rn_1j} > 0$ for some $j \in S$ and $\sum_{j \in S} q^*_{rn_2j} = 0$. Consider the coalition $N_n \setminus b_{rn_1}$ agreeing to the price vector $p = p^*$ and the trades q where $q_{ikj} = q^*_{ikj}$ for all $ik \neq rn_2$ and $j \in S$, and $q_{rn_2j} = q^*_{rn_1j}$ for all $j \in S$. By agreeing to the pair (p, q) the coalition $N_n \setminus b_{rn_1}$ can strictly increase the payoff to b_{rn_2} while leaving the payoffs to every other member of the coalition unchanged, contradicting the assumption that $x(p^*, q^*)$ lies in the core of (N_n, V).

Step 2. Now suppose to the contrary that $x(p^*, q^*)$ lies in the core of (N_n, V), and (D.19) does not hold for some $i \in B$ for whom there exists k and j such that $q^*_{ikj} > 0$. Select $m \in \{1, \ldots, n\}$. We can redistribute the quantities $\{q^*_{imj}\}_{j \in S}$ between the buyers $\{b_{i1}, \ldots, b_{in}\} \setminus b_{im}$ in such a way that quantities are only added to buyers for whom,

$$\sum_{j \in S} q^*_{ikj} < \frac{1}{n} D_i(\hat{p}^*), \tag{D.21}$$

and then only up to the point at which (D.21) holds with equality. The the failure of (D.19) to hold guarantees that there will be sufficient excess demand amongst the buyers in $\{b_{i1}, \ldots, b_{in}\} \setminus b_{im}$ to completely reassign the quantity $\sum_{j \in S} q^*_{imj}$ at the price $\hat{p}^*$ without violating rational rationing. Moreover, the members of the coalition $N_n \setminus b_{im}$ are no worse off, and any buyer who is allocated a portion of $\sum_{j \in S} q^*_{imj}$ receives a strictly greater payoff as a result of the deviation: A contradiction.

Step 3. Suppose to the contrary that for all $\underline{n} \in \mathbb{N}$ there exists $n \geq \underline{n}$, $q^* \in Q^r(p^*, N_n)$ and $i \in B$ such that $D_i(\hat{p}^*) > 0$, (D.20) holds with equality and $x(p^*, q^*)$ lies in the core of (N_n, V).

Set $\underline{n}$ such that,

$$\frac{1}{\underline{n}} D_l(\hat{p}^*) \leq D_i(\hat{p}^*), \tag{D.22}$$

for all $l \in B$. It is straightforward to see that at least one buyer $b_{hk} \in B_n$ must trade if value can be created. For all $n \geq \underline{n}$ we can redistribute the quantities $\{q^*_{hkj}\}_{j\in S}$ equally amongst the members of $\{b_{i1}, \ldots, b_{in}\}$. The members of the coalition $N_n \setminus b_{hk}$ are no worse off, and all buyers in $\{b_{i1}, \ldots, b_{in}\}$ receive a strictly greater payoff as a result of the deviation: A contradiction. $\square$

Lemma 10. *Suppose that buyer payoffs satisfy* (3.1) *with D_i twice continuously differentiable and strictly downward sloping for all $i \in B$. Suppose further that $|S| \geq 2$ and that $C'_j \geq 0$, $C''_j \geq 0$, $Y_j(p^c_N) > 0$ and $p^c_N \neq p^c_{N-j}$ for all $j \in S$.*

Let $p^*_j = \hat{p}^*$ for all $j \in S$. If $D_N(\hat{p}^*) < Y_N(\hat{p}^*)$ then there exists $n \in \mathbb{N}$ such that for all $q^* \in Q^r(p^*, N_n)$, $x(p^*, q^*) \notin \text{Core}(N_n, V)$.

Proof. Suppose to the contrary that $D_N(\hat{p}^*) < Y_N(\hat{p}^*)$ and for all $n \in \mathbb{N}$ there exists a $q^* \in Q^r(p^*, N_n)$ such $x(p^*, q^*) \in \text{Core}(N_n, V)$. First note that it must be the case that $\hat{p}^* > p^c_N$. Select $n \in \mathbb{N}$ such that,

$$\frac{1}{n} D_N(\hat{p}^*) < \frac{1}{|S|}\big(Y_N(\hat{p}^*) - D_N(\hat{p}^*)\big). \tag{D.23}$$

There must exist $j \in S$ such that,

$$Y_j(\hat{p}^*) - \sum_{\substack{i\in B\\ k\in\{1,\ldots,n\}}} q^*_{ikj} > \frac{1}{|S|}\big(Y_N(\hat{p}^*) - D_N(\hat{p}^*)\big). \tag{D.24}$$

It follows from the choice of n that there exists $m \in \mathbb{N}$ such that,

$$\sum_{\substack{i\in B\\ k\in\{1,\ldots,n\}}} q^*_{ikj} < \frac{m}{n} D_N(\hat{p}^*) \leq Y_j(\hat{p}^*). \tag{D.25}$$

It follows that the coalition $G = s_j \cup \{b_{i1}, \ldots, b_{im}\}_{i\in B}$ can deviate, agreeing to the price $\hat{p}^*$ and the quantities $q_{ik} = D_{ik}(\hat{p}^*)$ for all $i \in B$ and $k \in \{1, \ldots, m\}$. The deviation leaves all buyers in G at least as well off and strictly increase s_j's payoff: A contradiction. $\square$

E Proof of Proposition 4

Again we call on lemma 1 to provide the proof. Define $w : 2^N \longrightarrow \mathbb{R}$, such that $w(G) = 0$ for any coalition that does not contain at least one buyer, one intermediary and one seller, and,

$$w(G) = \int_0^{q_G^c} \left(D_G^{-1}(\vartheta) - T_G^{-1}(\vartheta) - Y_G^{-1}(\vartheta)\right) d\vartheta, \tag{E.1}$$

otherwise. Once again (N, V) and w satisfy the relationship set out in (1.5).

Definition 5. Let x be an allocation in the game (N, w). The *effective markup* $\Delta\bar{p}_k$ of a intermediary $m_k \in G$ is defined as,

$$\Delta\bar{p}_k = \frac{x_k + \int_0^{T_j(p_G^d - p_G^u)} T_j^{-1}(\vartheta) d\vartheta}{T_j(p_G^d - p_G^u)}. \tag{E.2}$$

By construction,

$$\sum_{i \in B \cap G} D_i(p_G^d) = \sum_{j \in M \cap G} T_j(p_G^d - p_G^u) = \sum_{j \in S \cap G} Y_j(p_G^u), \tag{E.3}$$

which in turn implies,

$$\frac{\sum_{i \in B \cap G} D_i(p_G^d)\bar{p}_i}{D_G(p_G^d)} = \frac{\sum_{j \in S \cap G} Y_j(p_G^u)\bar{p}_j}{Y_G(p_G^u)} + \frac{\sum_{j \in M \cap G} T_k(p_G^d - p_G^u)\Delta\bar{p}_k}{T_G(p_G^d - p_G^u)}, \tag{E.4}$$

which is to say the weighted average of effective prices paid buy the buyers in G is equal to the sum of the weighted averages of the seller effective prices and intermediary effective markups.

Lemma 11. *Suppose that intermediary cost functions satisfy $C_j' \geq 0$ and $C_j'' \geq 0$. For all $m_k \in M$ and $q \in \mathbb{R}_+$,*

$$T_j(p_N^d - p_N^u) \cdot (p_N^d - p_N^u) - \int_0^{T_j(p_N^d - p_N^u)} T_j^{-1}(\vartheta) d\vartheta < q\Delta\bar{p}_j - \int_0^q T_j^{-1}(\vartheta) d\vartheta, \tag{E.5}$$

implies $\Delta\bar{p}_j > p_N^d - p_N^u$.

Proof. Follows directly from the proof of lemma 3. □

Lemma 12. *Suppose that buyer payoffs satisfy* (3.1)*, and that seller and intermediary cost functions satisfy $C_j' \geq 0$ and $C_j'' \geq 0$. The allocation x^* is in* Core(N, w) *if the corresponding effective unit price $\bar{p}_i^* = p_N^d$ for all $i \in B$, $\Delta\bar{p}_j = p_N^d - p_N^u$ for all $k \in M$ and $\bar{p}_j^* = p_N^u$ for all $j \in S$.*

Proof. Buy way of contradiction suppose that there exists a coalition G such that,

$$\sum_{i \in G} x_i^* < w(G). \tag{E.6}$$

Let x be an allocation satisfying $\sum_{i \in G} x_i = w(G)$ and $x_i > x_i^*$ for all $i \in G$. Moreover define $\bar{p}_i$ (or $\Delta\bar{p}_i$) as the effective unit price (or markup) corresponding to the payoff x_i.

From lemma 3 we know that $\bar{p}_i < \bar{p}_i^* = p_N^d$ for all $i \in B \cap G$, from lemma 4 we know that $\bar{p}_j > \bar{p}_j^* = p_N^u$ for all $i \in S \cap G$, and $\Delta\bar{p}_k < \Delta\bar{p}_k^* = p_N^d - p_N^u$ for all $k \in M \cap G$. This implies,

$$\begin{aligned} \frac{\sum_{i \in B \cap G} D_i(p_G^d)\bar{p}_i}{D_G(p_G^d)} &< p_N^d \\ &< \frac{\sum_{j \in S \cap G} Y_j(p_G^u)\bar{p}_j}{Y_G(p_G^u)} + \frac{\sum_{j \in M \cap G} T_k(p_G^d - p_G^u)\Delta\bar{p}_k}{T_G(p_G^d - p_G^u)}, \end{aligned} \tag{E.7}$$

contradicting (E.4). □

F Proof of Proposition 5

Given rational rationing, the allocation $x(\hat{p}, \hat{q})$ can only be blocked by the coalition N. Consider the allocation $x(p, q)$ with $q = \min\{q_m^*, q_s^*\}$, $p^d = D^{-1}(q)$ and $p^u \in \left[Y^{-1}(q), p^d - T^{-1}(q)\right]$; there are two possibilities $\min\{q_m^*, q_s^*\} = q_m^*$ and $\min\{q_m^*, q_s^*\} = q_s^*$. In either case $x_b(p, q) > x_b(\hat{p}, \hat{q})$.

Suppose that $\min\{q_m^*, q_s^*\} = q_m^*$. Note that,

$$\hat{q}\big(\hat{p}^d - Y^{-1}(\hat{q})\big) - \int_0^{\hat{q}} T^{-1}(\vartheta)d\vartheta < \big(p^d - Y^{-1}(q)\big)q - \int_0^{q} T^{-1}(\vartheta)d\vartheta, \tag{F.1}$$

by assumption and,

$$\hat{q}Y^{-1}(\hat{q}) - \int_0^{\hat{q}} Y^{-1}(\vartheta)d\vartheta \le Y^{-1}(q)q - \int_0^{q} Y^{-1}(\vartheta)d\vartheta. \tag{F.2}$$

It follows that there is sufficient value at the pair (p, q) to leave both the seller and intermediary better off, the only remaining question is whether this value can be allocated in such a way as to improve the payoff to both seller and intermediary. If,

$$x_s(\hat{p},\hat{q}) \leq Y^{-1}(q)q - \int_0^q Y^{-1}(\vartheta)d\vartheta, \tag{F.3}$$

then set $p^u = Y^{-1}(q)$ and from (4.9) we can see that $x_m(\hat{p},\hat{q}) < x_m(p,q)$. For the case in which,

$$x_s(\hat{p},\hat{q}) > Y^{-1}(q)q - \int_0^q Y^{-1}(\vartheta)d\vartheta, \tag{F.4}$$

it follows from the concavity of (4.8) and the fact that $q_s^* \geq q > \hat{q}$, that,

$$x_s(\hat{p},\hat{q}) < \big(p^d - T^{-1}(q)\big)q - \int_0^q Y^{-1}(\vartheta)d\vartheta. \tag{F.5}$$

Therefore we can select $p^u \in \big[Y^{-1}(q), p^d - T^{-1}(q)\big]$ such that $x_s(p,q) = x_s(\hat{p},\hat{q})$ which leaves $x_m(p,q) > x_m(\hat{p},\hat{q})$.

If $\min\{q_m^*, q_s^*\} = q_s^*$ the same result can be developed by reversing the roles of the seller and intermediary.

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